Transverse Particle Dynamics*

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USPAS: "Beam Physics with Intense Space-Charge"
UCB: "Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

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Transverse Particle Dynamics: Outline

Transverse Particle Dynamics: Detailed Outline

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S1: Particle Equations of Motion

S1A: Introduction: The Lorentz Force Equation

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\boxed{\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]}$$

$m_i,\ \ q_i$ .... particle mass, charge $i =$ particle index

$\mathbf{x}_i(t)$ .... particle coordinate $t =$ time

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

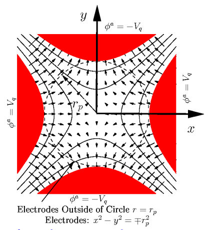
$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

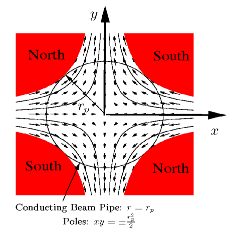
S1B: Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse Focusing Optics for focusing:

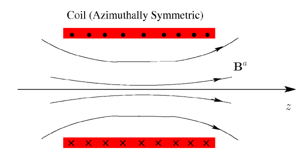
Electric Quadrupole



Magnetic Quadrupole

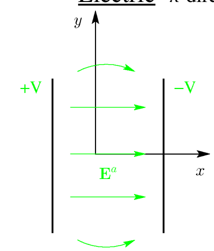


Solenoid

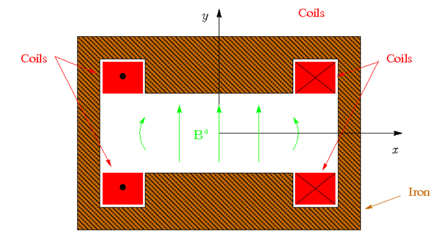


Dipole Bends:

Electric x-direction bend

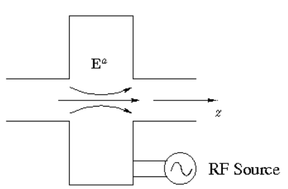


Magnetic x-direction bend

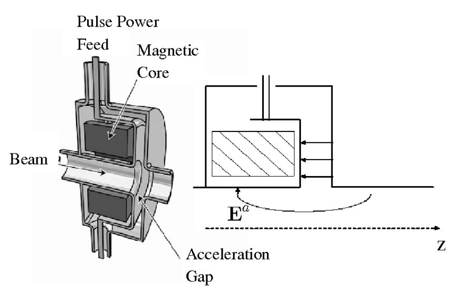


Longitudinal Acceleration:

RF Cavity



Induction Cell

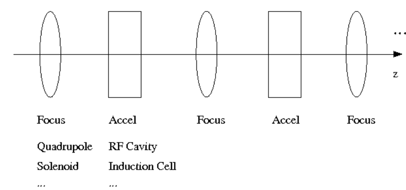


We will cover primarily transverse dynamics. Lectures by J.J. Barnard will cover acceleration and longitudinal physics:

Acceleration influences transverse dynamics – not possible to fully decouple

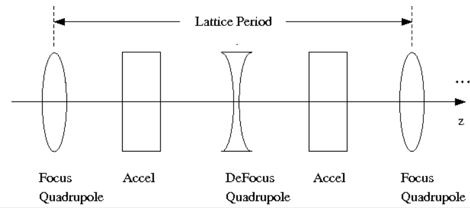
S1C: Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

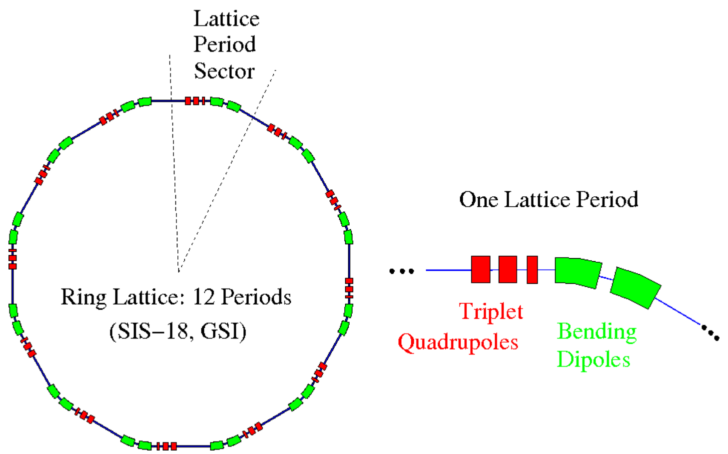


Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:

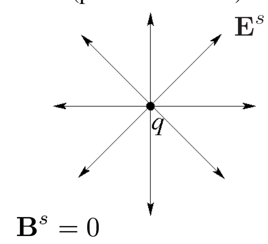


Lattices to insert beam into and out of ring further complicate lattice
Acceleration cells also present
(typically several RF cavities at one or more location)

S1D: Self fields

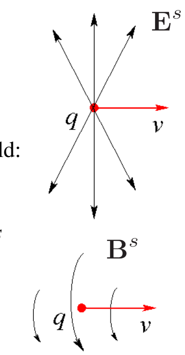
Self-fields are generated by the distribution of beam particles:
- Charges
- Currents

Particle at Rest
(pure electrostatic)



Particle in Motion

Obtain from Lorentz boost of rest-frame field: see Jackson, *Classical Electrodynamics*



Superimpose for all particles in the beam distribution
Accelerating particles also radiate
- We neglect electromagnetic radiation in this class (see: J.J. Barnard, Intro Lectures)

The electric ($\mathbf{E}^a$) and magnetic ($\mathbf{B}^a$) fields satisfy the Maxwell Equations. The linear structure of the Maxwell equations can be exploited to resolve the field into Applied and Self-Field components:

$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$

$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied Fields (often quasi-static) $\mathbf{E}^a, \ \mathbf{B}^a$
Generated by elements in lattice

$$\nabla \cdot \mathbf{E}^a = \frac{\rho^a}{\epsilon_0} \qquad \nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}^a + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a$$

$$\nabla \times \mathbf{E}^a = -\frac{\partial}{\partial t}\mathbf{B}^a \qquad \nabla \cdot \mathbf{B}^a = 0$$

$\rho^a =$ applied charge density $\qquad \dfrac{1}{\mu_0 \epsilon_0} = c^2$
$\mathbf{J}^a =$ applied current density

$+$ Boundary Conditions on $\mathbf{E}^a$ and $\mathbf{B}^a$

Boundary conditions depend on the total fields $\mathbf{E}, \ \mathbf{B}$
and if separated into Applied and Self-Field components, care can be required
System often solved as static boundary value problem and source free in the vacuum transport region of the beam

/// Aside: Notation:

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Cartesian Representation

$$= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Cylindrical Representation $\quad x = r\cos\theta, \quad y = r\sin\theta$

$$= \frac{\partial}{\partial \mathbf{x}}$$ - Abbreviated Representation

$$= \frac{\partial}{\partial \mathbf{x}_\perp} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Resolved Abbreviated Representation
Resolved into Perpendicular ($\perp$) and Parallel (z) components

$$\mathbf{x} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$

$$= \mathbf{x}_\perp + \hat{\mathbf{z}}z \qquad \mathbf{x}_\perp \equiv \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$$

In integrals, we denote:

$$\int d^3x \ \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \ \cdots = \int d^2x_\perp \int_{-\infty}^{\infty} dz \ \cdots$$

$$\int d^2x_\perp \ \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \ \cdots = \int_0^{\infty} dr\, r \int_{-\pi}^{\pi} d\theta \ \cdots$$

///

Self-Fields (dynamic, evolve with beam)

Generated by particle of the beam rather than (applied) sources outside beam

$$\nabla \cdot \mathbf{E}^s = \frac{\rho^s}{\epsilon_0} \qquad \nabla \times \mathbf{B}^s = \mu_0 \mathbf{J}^s + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^s$$

$$\nabla \times \mathbf{E}^s = -\frac{\partial}{\partial t}\mathbf{B}^s \qquad \nabla \cdot \mathbf{B}^s = 0$$

$\rho^s =$ beam charge density

$$= \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$\mathbf{J}^s =$ beam current density

$$= \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$i =$ particle index (N particles)

$q_i =$ particle charge

$\mathbf{x}_i =$ particle coordinate

$\mathbf{v}_i =$ particle velocity

$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$

$\delta(x) \equiv$ Dirac-delta function

$\sum_{i=1}^{N} \cdots =$ sum over beam particles

$+$ Boundary Conditions on $\mathbf{E}^s$ and $\mathbf{B}^s$ from material structures, radiation conditions, etc.

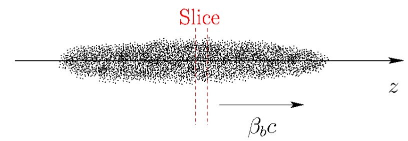
In accelerators, typically there is ideally a single species of particle:

$$\boxed{\begin{aligned} q_i &\rightarrow q \\ m_i &\rightarrow m \end{aligned}}$$

Large Simplification!

Multi-species results in more complex collective effects

Motion of particles within axial slices of the "bunch" are highly directed:



$$\beta_b(z) c \equiv \frac{1}{N'} \sum_{i=1}^{N'} \mathbf{v}_i \cdot \hat{\mathbf{z}}$$

$=$ Mean axial velocity of N' particles in beam slice

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t) = \hat{\mathbf{z}}\beta_b(z)c + \delta\mathbf{v}_i$$

$|\delta\mathbf{v}_i| \ll |\beta_b| c$ Paraxial Approximation

There are typically many particles:

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \simeq \rho(\mathbf{x}, t) \quad \text{continuous charge-density}$$

$$\mathbf{J}^s = \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)] \simeq \beta_b c \rho(\mathbf{x}, t)\hat{\mathbf{z}} \quad \text{continuous axial current-density}$$

The beam evolution is typically sufficiently slow (for heavy ions) where we can neglect radiation and approximate the self-field Maxwell Equations as:

See: J. J. Barnard, Intro. Lectures: Electrostatic Approximation

$$\mathbf{E}^s = -\nabla\phi$$

$$\mathbf{B}^s = \nabla \times \mathbf{A} \qquad \mathbf{A} = \hat{\mathbf{z}}\frac{\beta_b}{c}\phi$$

$$\nabla^2\phi = \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}}\phi = -\frac{\rho^s}{\epsilon_0}$$

$+$ Boundary Conditions on ϕ

Vast Reduction of self-field model:

Approximation equiv to electrostatic interactions in frame moving with beam

But still complicated!

Resolve the Lorentz force acting on beam particles into Applied and Self-Field terms:

$$\mathbf{F}_i(\mathbf{x}_i, t) = q\mathbf{E}(\mathbf{x}_i, t) + q\mathbf{v}_i(t) \times \mathbf{B}(\mathbf{x}_i, t)$$

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^s$$
$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$
$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied:

$$\mathbf{F}_i^a = q\mathbf{E}_i^a + q\mathbf{v}_i \times \mathbf{B}_i^a$$

$\mathbf{E}^a(\mathbf{x}_i, t) \equiv \mathbf{E}_i^a$ etc.

Self-Field:

$$\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s$$

The self-field force can be simplified:

See also: J.J. Barnard, Intro. Lectures

Plug in self-field forms:

$$\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i \times \mathbf{B}_i^s$$

$$\simeq q\left[-\left.\frac{\partial\phi}{\partial \mathbf{x}}\right|_i + (\beta_b c\hat{\mathbf{z}} + \delta\mathbf{v}_i) \times \left.\left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i\right]$$

0 Neglect: Paraxial

$$\left.\cdots\right|_i \equiv \left.\cdots\right|_{\mathbf{x}=\mathbf{x}_i}$$

Resolve into transverse (x and y) and longitudinal (z) components and simplify:

$$\beta_b c\hat{\mathbf{z}} \times \left.\left(\frac{\partial}{\partial \mathbf{x}} \times \hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i = \beta_b^2 \hat{\mathbf{z}} \times \left.\left(\frac{\partial}{\partial \mathbf{x}_\perp} \times \hat{\mathbf{z}}\phi\right)\right|_i$$

$$= \beta_b^2 \hat{\mathbf{z}} \times \left.\left(\frac{\partial\phi}{\partial y}\hat{\mathbf{x}} - \frac{\partial\phi}{\partial x}\hat{\mathbf{y}}\right)\right|_i$$

$$= \beta_b^2 \left.\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}}\right)\right|_i$$

$$= \beta_b^2 \left.\frac{\partial\phi}{\partial \mathbf{x}_\perp}\right|_i$$

also

$$-\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_i = -\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i - \left.\frac{\partial\phi}{\partial\mathbf{z}}\right|_i\hat{\mathbf{z}}$$

Together, these results give:

$$\boxed{\mathbf{F}_i^s = \underbrace{-\frac{q}{\gamma_b^2}\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i}_{\text{Transverse}} \underbrace{-\ \hat{\mathbf{z}}q\left.\frac{\partial\phi}{\partial z}\right|_i}_{\text{Longitudinal}}}$$

$$\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}} \qquad \text{Axial relativistic gamma of beam}$$

Transverse and longitudinal forces have different axial gamma factors
factor in transverse forces shows the space-charge forces become weaker as axial beam kinetic energy increases
- Most important in low energy (nonrelativistic) beam transport
- Strong in injectors

/// Aside: Singular Self Fields

In *free space*, the beam potential generated from the singular charge density:

$$\rho^s = \sum_{i=1}^{N} q_i\delta[\mathbf{x}-\mathbf{x}_i(t)]$$

is

$$\phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{1}{|\mathbf{x}-\mathbf{x}_i|}$$

Thus, the force of a particle at $\mathbf{x}=\mathbf{x}_i$ is:

$$\mathbf{F}_i = -q\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_i = \frac{q^2}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{(\mathbf{x}_i-\mathbf{x}_j)}{|\mathbf{x}_i-\mathbf{x}_j|^{3/2}}$$

Which diverges due to the $i = j$ term. This divergence is essentially "erased" when the continuous charge density is applied:

$$\rho^s = \sum_{i=1}^{N} q_i\delta[\mathbf{x}-\mathbf{x}_i(t)] \longrightarrow \rho(\mathbf{x},t)$$

Effectively removes effect of collisions
See: J.J. Barnard, Intro Lectures for more details
- Find collisionless Vlasov model of evolution is often adequate ///

The particle equations of motion in $\mathbf{x}_i - \mathbf{v}_i$ phase-space variables become:
Separate parts of $q\mathbf{E}_i^a + q\mathbf{v}_i\times\mathbf{B}_i^a$ into transverse and longitudinal comp

Transverse

$$\frac{d}{dt}\mathbf{x}_{\perp i} = \mathbf{v}_{\perp i}$$

$$\frac{d}{dt}(m\gamma_i\mathbf{v}_{\perp i}) \simeq \underbrace{q\mathbf{E}_{\perp i}^a + q\beta_b c\hat{\mathbf{z}}\times\mathbf{B}_{\perp i}^a + qB_{zi}^a\mathbf{v}_{\perp i}\times\hat{\mathbf{z}}}_{\text{Applied}} \underbrace{-\ q\frac{1}{\gamma_b^2}\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i}_{\text{Self}}$$

Longitudinal

$$\frac{d}{dt}z_i = v_{zi}$$

$$\frac{d}{dt}(m\gamma_i v_{zi}) \simeq \underbrace{qE_{zi}^a - q(v_{xi}B_{yi}^a - v_{yi}B_{xi}^a)}_{\text{Applied}} \underbrace{-\ q\left.\frac{\partial\phi}{\partial z}\right|_i}_{\text{Self}}$$

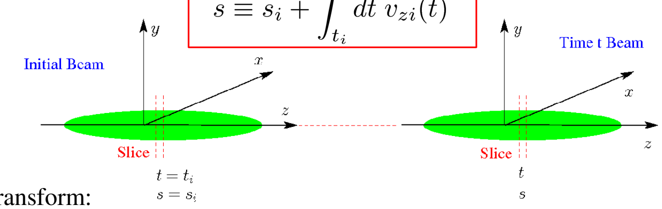
In the remainder of this (and most other) lectures, we analyze Transverse Dynamics. Longitudinal Dynamics will be covered in J.J. Barnard lectures
Except near injector, acceleration is typically slow
- Fractional change in γ_b, β_b small over characteristic transverse dynamical scales such as lattice period and betatron oscillation periods

Regard γ_b, β_b as specified functions given by the "acceleration schedule"

S1E: Equations of Motion in *s* and the Paraxial Approximation

In transverse accelerator dynamics, it is convenient to employ the axial coordinate (*s*) of a particle in the accelerator as the independent variable:

$$\boxed{s \equiv s_i + \int_{t_i}^{t} d\tilde{t}\ v_{zi}(\tilde{t})}$$



Transform:

$$v_{zi} = \frac{ds}{dt} \implies v_{xi} = \frac{dx_i}{dt} = \frac{ds}{dt}\frac{dx_i}{ds} = v_{zi}\frac{dx_i}{ds} = (\beta_b c + \delta v_{zi})\frac{dx_i}{ds}$$

Neglect

$$\simeq \beta_b c\frac{dx_i}{ds}$$

Denote:

$$\boxed{{}' \equiv \frac{d}{ds} \qquad \begin{aligned} v_{xi} &= \frac{dx_i}{dt} \simeq \beta_b c x_i' \\ v_{yi} &= \frac{dy_i}{dt} \simeq \beta_b c y_i' \end{aligned}}$$

Neglecting term consistent with assumption of small longitudinal momentum spread (paraxial approximation)

Procedure becomes more complicated when bends present: see S1H

In the paraxial approximation, x' and y' can be interpreted as the (small magnitude) angles that the particles make the with the z-axis:

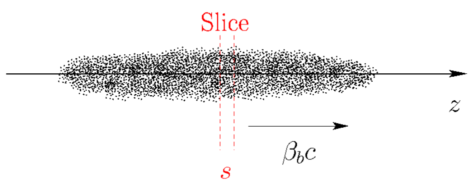
$$\boxed{\begin{aligned} x-\text{angle} &= \frac{v_{xi}}{v_{zi}} \simeq \frac{v_{xi}}{\beta_b c} = x'_i \\ y-\text{angle} &= \frac{v_{yi}}{v_{zi}} \simeq \frac{v_{yi}}{\beta_b c} = y'_i \end{aligned}}$$

Typical machine values:
$|x'| < 50$ mrad

The angles will be *small* in the paraxial approximation:

$$v_{xi}^2,\ v_{yi}^2 \ll \beta_b^2 c^2 \quad \Longrightarrow \quad x_i'^2,\ y_i'^2 \ll 1$$

Since the spread of axial momentum/velocities is small in the paraxial approximation, a thin axial slice of the beam maps to a thin axial slice and s can also be thought of as the axial coordinate of the slice in the accelerator lattice



$$\beta_b \equiv \sum_{i=1}^{N'} \frac{v_{zi}}{c}$$
slice

$$\boxed{s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\ \beta_b(\tilde{t})}$$

$$\boxed{s \simeq s_i + \int_{t_i}^{t} d\tilde{t}\ \beta_b(\tilde{t})}$$

The coordinate s can alternatively be interpreted as the axial coordinate of a reference (design) particle moving in the lattice.

It is desirable to express the particle equations of motion in terms of s rather than the time t

Makes it clear where you are in the lattice of the machine

Transform transverse particle equations of motion to s rather than t derivatives

$$\underbrace{\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i})}_{\text{Term 1}} \simeq q\mathbf{E}^a_{\perp i} + q\beta_b c \hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + \underbrace{qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}}}_{\text{Term 2}} - q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\bigg|_i$$

Transform Terms 1 and 2 in the particle equation of motion:

$$\frac{d}{dt} = v_{zi}\frac{d}{ds}$$

Term 1:
$$\begin{aligned} \frac{d}{dt}\left(m\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) &= m v_{zi}\frac{d}{ds}\left(\gamma_i v_{zi}\frac{d}{ds}\mathbf{x}_{\perp i}\right) \\ &= \underbrace{m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i}}_{\text{Term 1A}} + \underbrace{m v_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi})}_{\text{Term 1B}} \end{aligned}$$

Approximate:

Term 1A: $$m\gamma_i v_{zi}^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} \simeq m\gamma_b \beta_b^2 c^2 \frac{d^2}{ds^2}\mathbf{x}_{\perp i} = m\gamma_b\beta_b^2 c^2 \mathbf{x}''_{\perp i}$$

Term 1B: $$\begin{aligned} m v_{zi}\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_i v_{zi}) &\simeq m\beta_b c\left(\frac{d}{ds}\mathbf{x}_{\perp i}\right)\frac{d}{ds}(\gamma_b\beta_b c) \\ &\simeq m\beta_b c^2 (\gamma_b\beta_b)'\mathbf{x}'_{\perp i} \end{aligned}$$

Using the approximations 1A and 1B gives for Term 1:

$$\boxed{m\frac{d}{dt}\left(\gamma_i \frac{d\mathbf{x}_{\perp i}}{dt}\right) \simeq m\gamma_b\beta_b^2 c^2 \left[\mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i}\right]}$$

Similarly we approximate in Term 2:

$$\boxed{qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}} \simeq qB^a_{zi}\beta_b c \mathbf{x}'_{\perp i} \times \hat{\mathbf{z}}}$$

Using the simplified expressions for Terms 1 and 2 obtain the reduced transverse equation of motion:

$$\boxed{\begin{aligned} \mathbf{x}''_{\perp i} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_{\perp i} &= \frac{q}{m\gamma_b\beta_b^2 c^2}\mathbf{E}^a_{\perp i} + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} \\ &+ \frac{qB^a_{zi}}{m\gamma_b\beta_b c}\mathbf{x}'_{\perp i} \times \hat{\mathbf{z}} - \frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\bigg|_i \end{aligned}}$$

Will be analyzed extensively in lectures that follow in various limits to better understand solution properties

S1F: Axial Particle Kinetic Energy

Relativistic particle kinetic energy is:

$$\boxed{\mathcal{E} = (\gamma - 1)mc^2}$$

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

$$\mathbf{v} = (\beta_b + \delta\beta_z)c\hat{\mathbf{z}} + \beta_\perp c$$
$$= \text{Particle Velocity (3D)}$$

For a directed paraxial beam with motion primarily along the machine axis the kinetic energy is essentially the axial kinetic energy $\mathcal{E}_b$:

$$\mathcal{E} = (\gamma_b - 1)mc^2 \; + \; \Theta\left(\frac{|\delta\beta_z|}{\beta_b}, \frac{\beta_\perp^2}{\beta_b^2}\right)$$

$$\boxed{\mathcal{E} \simeq \mathcal{E}_b \equiv (\gamma_b - 1)mc^2}$$

In nonrelativistic limit: $\beta_b^2 \ll 1$

$$\boxed{\begin{aligned}\mathcal{E}_b \equiv (\gamma_b - 1)mc^2 &= \frac{1}{2}m\beta_b^2c^2 \; + \; \frac{3}{8}m\beta_b^4c^2 \; + \; \cdots \\ &\simeq \frac{1}{2}m\beta_b^2c^2 \; + \; \Theta(\beta_b^4)\end{aligned}}$$

Convenient units:

Electrons:

$$\boxed{m = m_e = 511 \; \frac{\text{keV}}{c^2}}$$

Electrons rapidly relativistic due to relatively low mass

Ions/Protons:

$$\boxed{m = (\text{atomic mass}) \cdot m_u \qquad \begin{aligned} m_u &\equiv \text{Atomic Mass Unit} \\ &= 931.49 \; \frac{\text{MeV}}{c^2}\end{aligned}}$$

Note:

$$m_p = \text{Proton Mass} = 938.27 \; \frac{\text{MeV}}{c^2}$$

$$m_n = \text{Neuton Mass} = 939.57 \; \frac{\text{MeV}}{c^2}$$

$$m_p \simeq m_n \simeq 940 \frac{\text{MeV}}{c^2}$$

Approximate roughly for ions:

$$\boxed{m \simeq Am_u \qquad \begin{aligned} A = &\text{Mass Number} \\ &\text{(Number Nucleons)}\end{aligned}}$$

$m_u \gg m_e$

Protons/ions take much longer to become relativistic than electrons

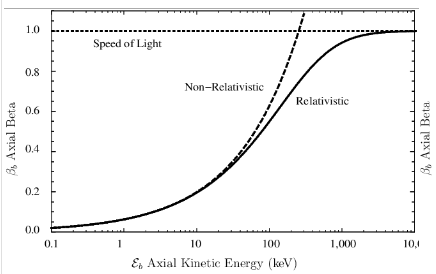
$m_p, m_u > m_u$ due to nuclear binding energy

$$\frac{\mathcal{E}_b/A}{m_u c^2} \simeq \gamma_b - 1 \qquad \Longrightarrow \qquad \boxed{\begin{aligned}\gamma_b &= 1 + \frac{\mathcal{E}_b/A}{m_u c^2} \\ \beta_b &= \sqrt{1 - 1/\gamma_b^2}\end{aligned}}$$

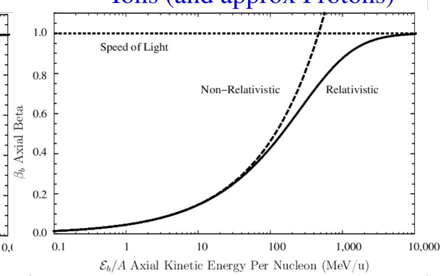
Energy/Nucleon $\mathcal{E}_b/A$ fixes β_b to set phase needs of RF cavities

Contrast beam relativistic β_b for electrons and protons/ions:

Electrons



Ions (and approx Protons)



Notes: 1) plots do to overlay, scale changed
2) Ion plot slightly off for protons since $m_u \neq m_p$

- Electrons become relativistic easier relative to protons/ions due to light mass
- Space-charge more important for ions than electrons (see Sec. S1D)
  - Low energy ions near injector expected to have strongest space-charge

S1G: Summary: Transverse Particle Equations of Motion

$$\boxed{\begin{aligned}
&\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp' \times \hat{\mathbf{z}} \\
&\qquad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}_\perp}\phi \\
&\mathbf{E}^a = \text{Applied Electric Field} \qquad {}' \equiv \frac{d}{ds} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}} \\
&\mathbf{B}^a = \text{Applied Magnetic Field} \\
&\nabla^2\phi = \frac{\partial}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{x}}\phi = -\frac{\rho}{\epsilon_0} \\
&+ \text{Boundary Conditions on } \phi
\end{aligned}}$$

Drop particle *i* subscripts (in most cases) henceforth to simplify notation

Neglects axial energy spread, bending, and electromagnetic radiation

$\gamma-$factors different in applied and self-field terms:

In $-\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}}\phi$, contibutions to γ_b^3:

$\gamma_b \Longrightarrow$ Kinematics

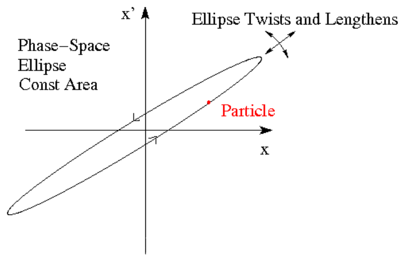
$\gamma_b^2 \Longrightarrow$ Self-Magnetic Field Corrections (leading order)

S1H: Preview: Analysis to Come

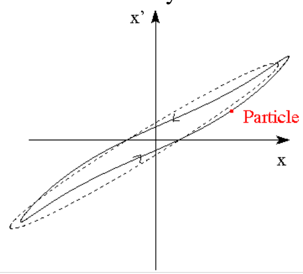
Much of transverse accelerator physics centers on understanding the evolution of beam particles in 4-dimensional *x-x'* and *y-y'* phase space.

Typically, restricted 2-dimensional phase-space projections in *x-x'* and/or *y-y'* are analyzed to simplify interpretations:

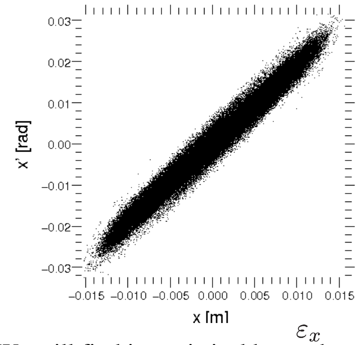
When forces are linear particles tend to move on ellipses of constant area
- Ellipse may elongate/shrink and rotate as beam evolves in lattice



Nonlinear force components distort orbits and cause undesirable effects
- Growth in effective phase-space area reduces focusability



The "effective" phase-space volume of a distribution of beam particles is of fundamental interest



Effective area measure in *x-x'* phase-space is the *x*-emittance

Statistical "Area" $\sim \pi\varepsilon_x$

$$\varepsilon_x = 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

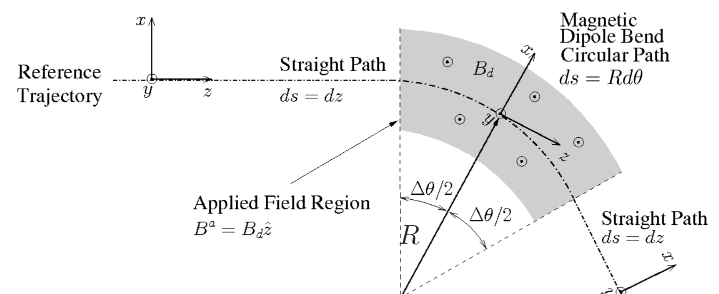
We will find in statistical beam descriptions that:

Larger/Smaller beam phase-space areas (Larger/Smaller emittances) $\Longleftrightarrow$ Harder/Easier to focus beam on <u>small</u> final spots

Much of advanced accelerator physics centers on preserving beam quality by understanding and controlling emittance growth due to nonlinear forces arising from both space-charge and the applied focusing. In the remainder of the next few lectures we will review the physics of transverse particle dynamics of single particles moving in linear applied fields. Later, we will generalize concepts to include forces from space-charge in this formulation and nonlinear effects from both applied and self-fields.

S1I: Bent Coordinate System and Particle Equations of Motion with Dipole Bends and Axial Momentum Spread

The previous equations of motion can be applied to dipole bends provided the *x,y,z* coordinate system is fixed. In practice, it can prove more convenient to employ coordinates that follow the beam in a bend.



$\Theta =$ Bend Angle

$R =$ Bend Radius

$s =$ Reference Trajectory Coordinate

In this perspective, dipoles are adjusted given the design momentum of the reference particle to bend the orbit through a radius *R*.

- Bends usually only in one plane (say *x*)
  - Implemented by a dipole applied field: E^a_x or B^a_y
- Easy to apply material analogously for *y*-plane bends, if necessary

Denote:

$$\boxed{p_0 = m\gamma_b\beta_b c = \text{design momentum}}$$

Then a magnetic *x*-bend through a radius *R* is specified by:

$$\mathbf{B}^a = B^a_y\hat{\mathbf{y}} = \text{const in bend}$$
$$\frac{1}{R} = \frac{qB^a_y}{p_0}$$

Analogous formula for Electric Bend will be derived in problem set

The particle rigidity is defined as ($[B\rho]$ read as one symbol called "B-Rho"):

$$\boxed{[B\rho] \equiv \frac{p_0}{q} = \frac{m\gamma_b\beta_b c}{q}}$$

is often applied to express the bend result as:

$$\boxed{\frac{1}{R} = \frac{B^a_y}{[B\rho]}}$$

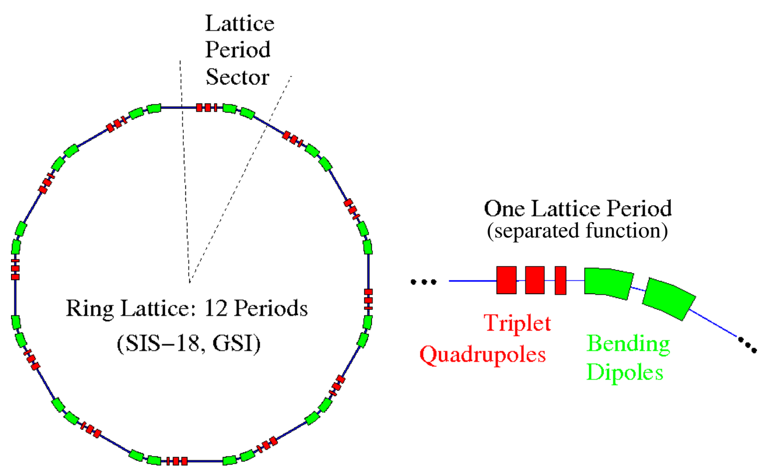
Comments on bends:

- *R* can be positive or negative depending on sign of $B^a_y/[B\rho]$
- For straight sections, $R \rightarrow \infty$ (or equivalently, $B^a_y = 0$)
- Lattices often made from discrete element dipoles and straight sections with separated function optics
  - Bends sometimes provide "edge focus" in a ring
  - Sometimes elements for bending/focusing are combined
- For a ring, dipoles strengths are tuned with particle rigidity/momentum so the reference orbit makes a closed path lap through the circular machine
  - Dipoles adjusted as particles gain energy to maintain closed path
  - In a Synchrotron dipoles and focusing elements are adjusted together to maintain focusing and bending properties as the particles gain energy. This is the origin of the name "Synchrotron."
- Total bending strength of a ring in Tesla-meters limits the ultimately achievable particle energy/momentum in the ring

/// Example: Typical separated function lattice in a Synchrotron

Focus Elements in Red

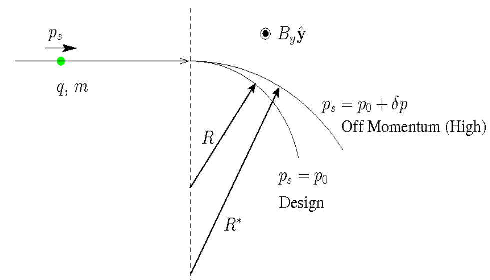
Bending Elements in Green



For "off-momentum" errors:

$$p_s = p_0 + \delta p$$
$$p_0 = m\gamma_b\beta_b c = \text{design momentum}$$
$$\delta p = \text{off- momentum}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii



- Not usual to have acceleration in bends
  - Dipole bends and quadrupole focusing are sometimes combined

Transverse particle equations of motion including "off-momentum" effects:

See texts such as Edwards and Syphers for guidance on derivation steps

Full derivation is beyond needs/scope of this class

$$\begin{aligned}
x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta}\right]x &= \frac{\delta}{1+\delta}\frac{1}{R(s)} + \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_x^a}{(1+\delta)^2} \\
&\quad - \frac{q}{m\gamma_b\beta_b c}\frac{B_y^a}{1+\delta} + \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial x} \\
y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' &= \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_y^a}{(1+\delta)^2} + \frac{q}{m\gamma_b\beta_b c}\frac{B_x^a}{1+\delta} \\
&\quad - \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial y}
\end{aligned}$$

$$p_0 = m\gamma_b\beta_b c = \text{Design Momentum} \qquad \frac{1}{R(s)} = \frac{B_y^a(s)|_{\text{Dipole}}}{[B\rho]} \qquad [B\rho] = \frac{p_0}{q}$$

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

Comments:

Design bends only in *x* and $B_y^a,\ E_x^a$ contain *no* dipole terms (design orbit)

- Dipole components set via the design bend radius *R*(*s*)

Equations contain only low-order terms in momentum spread δ

Comments continued:

Equations are often applied linearized in δ

Achromatic focusing lattices are often designed using equations with momentum spread to obtain focal points independent of δ to some order

x and *y* equations differ significantly due to bends modifying the *x*-equation when *R*(*s*) is finite

It will be shown in the problems that for electric bends:

$$\frac{1}{R(s)} = \frac{E_x^a(s)}{\beta_b c[B\rho]}$$

Applied fields for focusing: $\mathbf{E}_\perp^a,\ \mathbf{B}_\perp^a,\ B_s^a$ must be expressed in the bent *x,y,s* system of the reference orbit

- Includes error fields in dipoles

Self fields may also need to be solved taking into account bend terms

- Often can be neglected in Poisson's Equation

$$\left\{\frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial}{\partial x}\right] + \frac{\partial^2}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial}{\partial s}\right]\right\}\phi = -\frac{\rho}{\epsilon_0}$$

if $R \rightarrow \infty$ reduces to familiar:

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2}\right\}\phi = -\frac{\rho}{\epsilon_0}$$

Appendix A: Gamma and Beta Factor Conversions

It is frequently the case that relativistic gamma and beta factors must be converted when analyzing transverse particle dynamics. Here we summarize some useful formulas in that come up when comparing various forms of equations. Derivatives are taken wrt the axial coordinate *s* but also apply wrt time *t*

Results summarized here can be immediately applied in the paraxial approximation by taking:

$$v = |\mathbf{v}| \simeq v_b = \beta_b c \qquad \Longrightarrow \qquad \begin{aligned}\beta &\simeq \beta_b \\ \gamma &\simeq \gamma_b\end{aligned}$$

Assume that the beam is forward going with $\beta \geq 0$:

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \gamma\sqrt{\gamma^2-1}$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \beta^2 = 1 - 1/\gamma^2$$

A commonly occurring acceleration factor is:

$$\frac{(\gamma\beta)'}{(\gamma\beta)} = \frac{\gamma'}{\gamma} + \frac{\beta'}{\beta} = \frac{\gamma'}{\gamma\beta^2}$$

Axial derivative factors can be converted using:

$$\gamma' = \frac{\beta\beta'}{(1-\beta^2)^{3/2}} \qquad \beta' = \frac{\gamma'}{\gamma^2\sqrt{\gamma^2-1}}$$

S2: Transverse Particle Equations of Motion in Linear Applied Focusing Channels

S2A: Introduction

Write out transverse particle equations of motion in explicit component form:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' = \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Equations previously derived under assumptions:

- No bends (fixed *x*-*y*-*z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq$ const)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

The applied focusing fields

Electric:	$E_x^a,\ E_y^a\ E_z^a$
Magnetic:	$B_x^a,\ B_y^a\ B_z^a$

must be specified as a function of *s* and the transverse particle coordinates *x* and *y* to complete the description

- Consistent change in axial velocity ($\beta_b c$) due to E_z^a must be evaluated
  - Typically due to RF cavities and/or induction cells
- Restrict analysis to fields from applied focusing structures

Intense beam accelerators and transport lattices are designed to optimize *linear* applied focusing forces with terms:

Electric:

$$E_x^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$E_y^a \simeq (\text{function of } s) \times (x \text{ or } y)$$

Magnetic:

$$B_x^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$B_y^a \simeq (\text{function of } s) \times (x \text{ or } y)$$
$$B_z^a \simeq (\text{function of } s)$$

Common situations that realize these linear applied focusing forms will be overviewed:

- Continuous Focusing (see: S2B)
- Quadrupole Focusing
  - Electric (see: S2C)
  - Magnetic (see: S2D)
- Solenoidal Focusing (see: S2E)

Other situations that will not be covered (typically more nonlinear optics):

- Einzel Lens (see: J.J. Barnard, Intro Lectures)
- Plasma Lens
- Wire guiding

Why design around linear applied fields ?

- Linear oscillators have well understood physics allowing formalism to be developed that can guide design
- Linear fields are in some sense "lower order" so it should be possible for a given source amplitude field terms with greater strength than for "higher order" nonlinear fields

S2B: Continuous Focusing

Assume constant electric field applied focusing force:

$$\mathbf{B}^a = 0$$

$$\mathbf{E}_\perp^a = E_x^a\hat{\mathbf{x}} + E_y^a\hat{\mathbf{y}} = -\frac{m\gamma_b\beta_b^2c^2k_{\beta 0}^2}{q}\mathbf{x}_\perp \qquad k_{\beta 0}^2 \equiv \text{const} > 0$$

$$E_z^a = 0 \qquad [k_{\beta 0}] = \frac{\text{rad}}{\text{m}}$$

Continuous focusing equations of motion:

Insert field components into linear applied field equations and collect terms

$$\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' + k_{\beta 0}^2\mathbf{x}_\perp = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}$$

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + k_{\beta 0}^2 x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + k_{\beta 0}^2 x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

Equivalent Component Form

Even this simple model can become complicated

- Space charge: ϕ must be calculated consistent with beam evolution
- Acceleration: acts to damp orbits (see: S10)

Simple model in limit of no acceleration ($\gamma_b \beta_b \simeq \text{const}$) and negligible space-charge ($\phi \simeq \text{const}$):

$$\boxed{\mathbf{x}''_\perp + k_{\beta 0}^2 \mathbf{x}_\perp = 0} \implies \text{orbits simple harmonic oscillatons}$$

General solution is elementary:

$$\begin{aligned}
\mathbf{x}_\perp &= \mathbf{x}_\perp(s_i)\cos[k_{\beta 0}(s-s_i)] + [\mathbf{x}'_\perp(s_i)/k_{\beta 0}]\sin[k_{\beta 0}(s-s_i)] \\
\mathbf{x}'_\perp &= -k_{\beta 0}\mathbf{x}_\perp(s_i)\sin[k_{\beta 0}(s-s_i)] + \mathbf{x}'_\perp(s_i)\cos[k_{\beta 0}(s-s_i)]
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}_\perp(s_i) &= \text{Initial coordinate} \\
\mathbf{x}'_\perp(s_i) &= \text{Initial angle}
\end{aligned}$$

In terms of a transfer map in the *x*-plane (*y*-plane analogous):

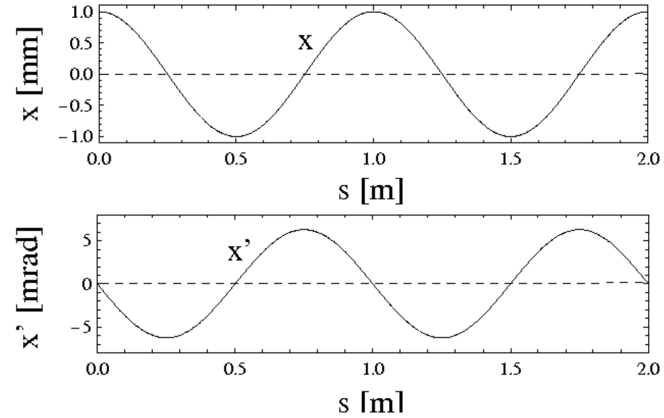
$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \mathbf{M}_x(s|s_i) \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_i}$$

$$\mathbf{M}_x(s|s_i) = \begin{bmatrix} \cos[k_{\beta 0}(s-s_i)] & \frac{1}{k_{\beta_0}}\sin[k_{\beta 0}(s-s_i)] \\ -k_{\beta 0}\sin[k_{\beta 0}(s-s_i)] & \cos[k_{\beta 0}(s-s_i)] \end{bmatrix}$$

/// Example: Particle Orbits in Continuous Focusing

Particle phase-space in *x*-*x'* with only applied field

$$k_{\beta 0} = 2\pi \text{ rad/m} \qquad x(0) = 1 \text{ mm}$$
$$\phi \simeq 0 \quad \gamma_b\beta_b = \text{const} \qquad x'(0) = 0$$



Orbits in the applied field are just simple harmonic oscillators ///

Problem with continuous focusing model:

The continuous focusing model is realized by a stationary ($m \rightarrow \infty$) partially neutralizing uniform background of charges filling the beam pipe. To see this apply Maxwell's equations to the applied field to calculate an applied charge density:

$$\boxed{\rho^a = \epsilon_0 \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{E}^a = -\frac{2m\epsilon_0\gamma_b\beta_b^2c^2k_{\beta 0}^2}{q} = \text{const}}$$

Unphysical model, but commonly employed since it represents the average action of more physical focusing fields in a simpler to analyze model

- Demonstrate later in simple examples and problems given

Continuous focusing can provide reasonably good estimates for more realistic periodic focusing models if $k_{\beta 0}^2$ is appropriately identified in terms of "equivalent" parameters *and* the periodic system is stable.

- See lectures that follow and homework problems for examples

In more realistic models, one requires that *quasi-static* focusing fields in the machine aperture satisfy the vacuum Maxwell equations

$$\begin{aligned}
\nabla \cdot \mathbf{E}^a &= 0 & \nabla \cdot \mathbf{B}^a &= 0 \\
\nabla \times \mathbf{E}^a &= 0 & \nabla \times \mathbf{B}^a &= 0
\end{aligned}$$

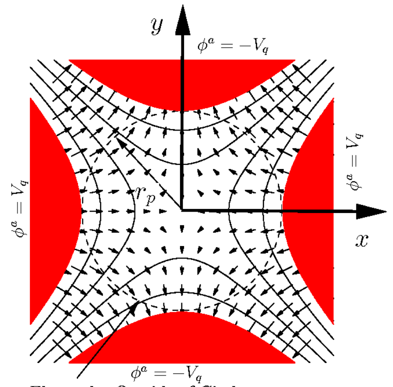
Require in the region of the beam
Applied field sources outside of the beam region

The vacuum Maxwell equations constrain the 3D form of applied fields resulting from spatially localized lenses. The following cases are commonly exploited to optimize linear focusing strength in physically realizable systems while keeping the model relatively simple:

1) Alternating Gradient Quadrupoles with transverse orientation
   - Electric Quadrupoles (see: S2C)
   - Magnetic Quadrupoles (see: S2D)
2) Solenoidal Magnetic Fields with longitudinal orientation (see: S2E)
3) Einzel Lenses (see J.J. Barnard, Introductory Lectures)

S2C: Alternating Gradient Quadrupole Focusing
Electric Quadrupoles

In the axial center of a long electric quadrupole, model the fields as 2D transverse



Electrodes Outside of Circle $r = r_p$
Electrodes: $x^2 - y^2 = \mp r_p^2$

Electrodes hyperbolic

Structure infinitely extruded along z

2D Transverse Fields

$$\mathbf{B}^a = 0$$

$$E_x^a = -Gx$$
$$E_y^a = Gy$$
$$E_z^a = 0$$

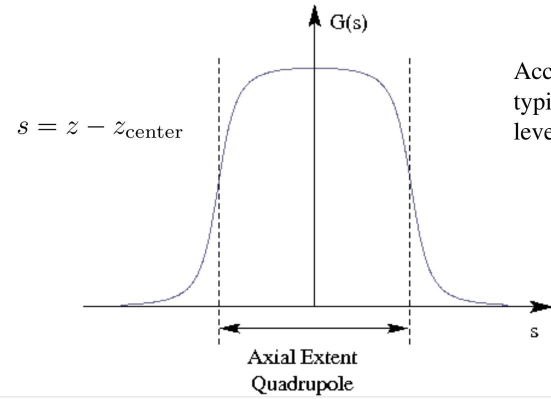
$$G \equiv \frac{2V_q}{r_p^2} = -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y}$$
$= \text{Electric Gradient}$

$V_q = \text{Pole Voltage}$

$r_p = \text{Pipe Radius}$ (clear aperture)

Quadrupoles actually have finite axial length in z. Model this by taking the gradient G to vary in s, i.e., $G = G(s)$ with $s = z - z_{\text{center}}$ (straight section)

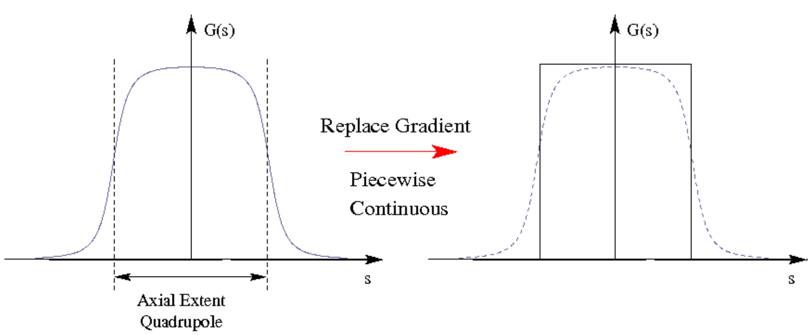
- Variation is called the fringe-field of the focusing element
- Variation will violate the Maxwell Equations in 3D
  - Provides a reasonable first approximation in many applications
- Usually quadrupole is long, and $G(s)$ will have a flat central region and rapid variation near the ends



Accurate fringe calculation typically requires higher level modeling:
- 3D analysis
- Detailed geometry

For many applications the actual quadrupole fringe function $G(s)$ is replaced by a simpler function to allow more idealized modeling

- Replacements should be made in an "equivalent" parameter sense to be detailed later (see: lectures on Transverse Centroid and Envelope Modeling)
- Fringe functions often replaced in design studies by piecewise constant $G(s)$
  - Commonly called "hard-edge" approximation
- See S3 and Lund and Bukh, PRSTAB 7 924801 (2004), Appendix C for more details on equivalent models



Electric quadrupole equations of motion:

Insert applied field components into linear applied field equations and collect terms

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \\ \kappa(s) = \frac{qG}{m\gamma_b\beta_b^2c^2} &= \frac{G}{\beta_b c[B\rho]} \\ G = -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2} \qquad & [B\rho] = \frac{m\gamma_b\beta_b c}{q} \end{aligned}}$$

- For positive/negative κ, the applied forces are Focusing/deFocusing in the x- and y-planes
- The x- and y-equations are decoupled
- Valid whether the the focusing function κ is piecewise constant or incorporates a fringe model

Simple model in limit of no acceleration ($\gamma_b \beta_b \simeq$ const) and negligible space-charge ($\phi \simeq$ const) and $\kappa =$ const:

$$\boxed{\begin{aligned} x'' + \kappa x &= 0 \\ y'' - \kappa y &= 0 \end{aligned}} \qquad \Longrightarrow \text{ orbits harmonic or hyperbolic depending on sign of } \kappa$$

General solution:

$\kappa > 0$:

$$x = x_i \cos[\sqrt{\kappa}(s - s_i)] + x_i'/\sqrt{\kappa} \sin[\sqrt{\kappa}(s - s_i)]$$

$$x' = -\sqrt{\kappa} x_i \sin[\sqrt{\kappa}(s - s_i)] + x_i' \cos[\sqrt{\kappa}(s - s_i)]$$

$$x(s_i) = x_i = \text{ Initial coordinate}$$

$$x'(s_i) = x_i' = \text{ Initial angle}$$

$$y = y_i \cosh[\sqrt{\kappa}(s - s_i)] + y_i'/\sqrt{\kappa} \sinh[\sqrt{\kappa}(s - s_i)]$$

$$y' = \sqrt{\kappa} x_i \sinh[\sqrt{\kappa}(s - s_i)] + y_i' \cosh[\sqrt{\kappa}(s - s_i)]$$

$$y(s_i) = y_i = \text{ Initial coordinate}$$

$$y'(s_i) = y_i' = \text{ Initial angle}$$

$\kappa < 0$:

Exchange x and y in $\kappa > 0$ case.

In terms of a transfer maps:

$\kappa > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \mathbf{M}_x(s|s_i) \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s_i}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = \mathbf{M}_y(s|s_i) \cdot \begin{pmatrix} y \\ y' \end{pmatrix}_{s_i}$$

$$\mathbf{M_x}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\kappa}(s - s_i) & \frac{1}{\sqrt{\kappa}} \sin[\sqrt{\kappa}(s - s_i) \\ -\sqrt{\kappa} \sin[\sqrt{\kappa}(s - s_i) & \cos[\sqrt{\kappa}(s - s_i) \end{bmatrix}$$

$$\mathbf{M_y}(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\kappa}(s - s_i) & \frac{1}{\sqrt{\kappa}} \sinh[\sqrt{\kappa}(s - s_i) \\ \sqrt{\kappa} \sinh[\sqrt{\kappa}(s - s_i) & \cosh[\sqrt{\kappa}(s - s_i) \end{bmatrix}$$

$\kappa < 0$:

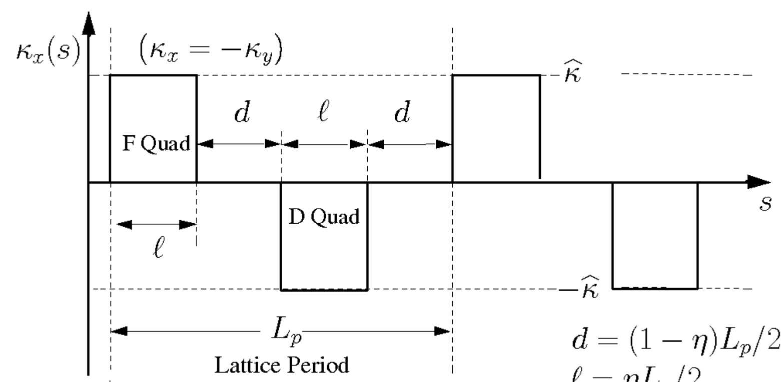
Exchange x and y in $\kappa > 0$ case.

Quadrupoles must be arranged in a lattice where the particles traverse a sequence of optics with alternating gradient to focus strongly in both transverse directions

- Alternating gradient necessary to provide focusing in both *x*- and *y*-planes
- Alternating Gradient Focusing often abbreviated "AG" and is sometimes called "Strong Focusing"
- Parameters should be tuned with particle properties and oscillation phases for proper operation
  - F (Focus) in plane placed where excursions (on average) are small
  - D (deFocus) placed where excursions (on average) are large
  - O (drift) allows axial separation between elements
- Focusing lattices often (but not necessarily) periodic
  - Periodic expected to give optimal efficiency in focusing with quadrupoles
- Drifts between F and D quadrupoles allow space for: acceleration cells, beam diagnostics, vacuum pumping, ...

Example Quadrupole FODO periodic lattices with piecewise constant κ

- FODO: [Focus drift(O) DeFocus Drift(O)] has equal length drifts and same length F and D quadrupoles
- FODO is simplest possible realization of "alternating gradient" focusing
- - Can also have thin lens limit of finite axial length magnets in FODO lattice



$$d = (1 - \eta) L_p/2$$

$$\ell = \eta L_p/2$$

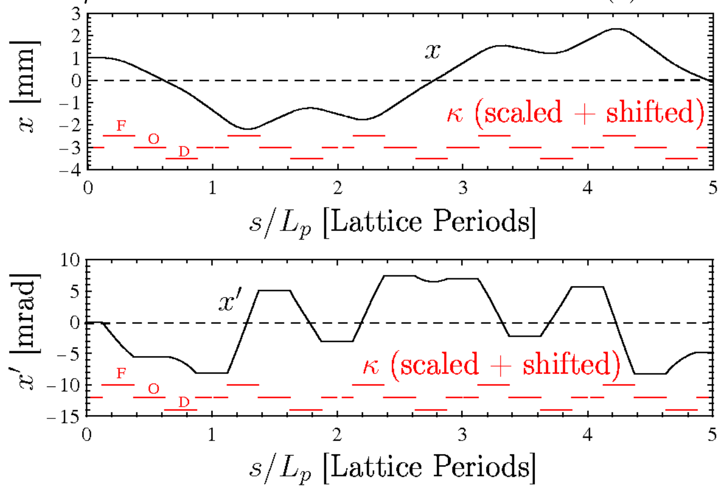
$$\eta = \text{Occupancy} \in (0, 1]$$

/// Example: Particle Orbits in a FODO Periodic Quadrupole Focusing Lattice:

Particle phase-space in x-x' with only hard-edge applied field

$L_p = 0.5$ m $\quad \kappa = \pm 50\ \mathrm{rad/m^2}$ in Quads $\quad x(0) = 1$ mm

$\eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b\beta_b = \mathrm{const} \quad x'(0) = 0$



///

Comments on Orbits:

Orbits strongly deviate from simple harmonic form due to AG focusing
- Multiple harmonics present

Orbit tends to be farther from axis in focusing quadrupoles and closer to axis in defocusing quadrupoles to provide net focusing

Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)

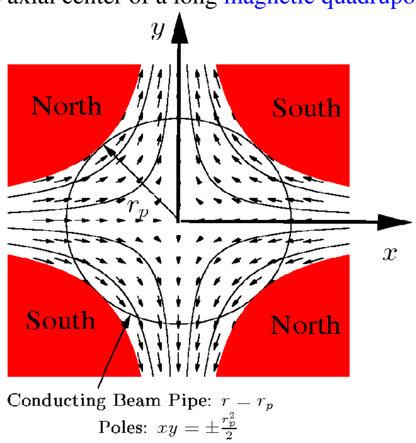
y-orbit has the same properties as x-orbit due to the periodic structure and AG focusing

If quadrupoles are rotated about their z-axis of symmetry, then the x- and y-equations become cross-coupled. This is called quadrupole skew coupling (see: Appendix A) and complicates the dynamics.

Some properties of particle orbits in quadrupoles with $\kappa = \mathrm{const}$ will be analyzed in the problem sets

S2D: Alternating Gradient Quadrupole Focusing Magnetic Quadrupoles

In the axial center of a long magnetic quadrupole, model fields as 2D transverse



Magnetic (ideal iron) poles hyperbolic

Structure infinitely extruded along z

2D Transverse Fields

$$\mathbf{E}^a = 0$$

$$B_x^a = Gy$$
$$B_y^a = Gx$$
$$B_z^a = 0$$

$$G \equiv \frac{B_q}{r_p} = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \text{Magnetic Gradient}$$

$$B_q = |\mathbf{B}^a|_{r=r_p} = \text{Pole Field}$$

$$r_p = \text{Pipe Radius}$$

Analogously to the electric quadrupole case, take $G = G(s)$

Same comments made on electric quadrupole fringe in S2C are directly applicable to magnetic quadrupoles

Magnetic quadrupole equations of motion:

Insert field components into linear applied field equations and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$\kappa(s) = \frac{qG}{m\gamma_b\beta_b c} = \frac{G}{[B\rho]}$$

$$G = \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_q}{r_p} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

Equations identical to the electric quadrupole case in terms of $\kappa(s)$

All comments made on electric quadrupole focusing lattice are immediately applicable to magnetic quadruples: just apply different κ definitions in design

Scaling of κ with energy different than electric case impacts applicability

$$\kappa = \begin{cases} \dfrac{G}{\beta_b c [B\rho]} & \text{Electric Focusing;} \quad G = \dfrac{\partial E_y^a}{\partial y} = \dfrac{2V_q}{r_p^2} \\ \dfrac{G}{[B\rho]} & \text{Magnetic Focusing;} \quad G = \dfrac{\partial B_x^a}{\partial y} = \dfrac{B_q}{r_p} \end{cases}$$

Electric focusing weaker for higher particle energy (larger β_b)
Technical limit values of gradients
- Voltage holding for electric
- Material properties (iron saturation, superconductor limits, ...) for magnetic

See JJB Intro lectures for discussion on focusing technology choices

Different energy dependence also gives different dispersive properties when beam has axial momentum spread:

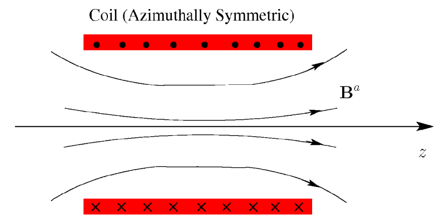
$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

$$\kappa \to \begin{cases} \frac{\kappa}{(1+\delta)^2} & \text{Electric Focusing} \\ \frac{\kappa}{1+\delta} & \text{Magnetic Focusing} \end{cases}$$

Electric case further complicated because δ couples to the transverse motion since particles crossing higher electrostatic potentials are accelerated/deaccelerated

S2E: Solenoidal Focusing

The field of an ideal magnetic solenoid is invariant under transverse rotations about it's axis of symmetry (z) can be expanded in terms of the on-axis field as as:



$$\mathbf{E}^a = 0$$

$$\mathbf{B}_\perp^a = \frac{1}{2} \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{|\mathbf{x}_\perp|}{2} \right)^{2\nu-2} \mathbf{x}_\perp$$

$$B_z^a = B_{z0}(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{|\mathbf{x}_\perp|}{2} \right)^{2\nu}$$

$$B_{z0}(z) \equiv B_z^a(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

See Reiser, *Theory and Design of Charged Particle Beams*, Sec. 3.3.1

For modeling, we truncate the expansion using only leading-order terms to obtain:
Corresponds to linear dynamics in the equations of motion

$$B_x^a = -\frac{1}{2} \frac{\partial B_{z0}(z)}{\partial z} x$$

$$B_y^a = -\frac{1}{2} \frac{\partial B_{z0}(z)}{\partial z} y \qquad B_{z0}(z) \equiv B_z^a(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

$$B_z^a = B_{z0}(z)$$

Note that this truncated expansion is divergence free:

$$\nabla \cdot \mathbf{B}^a = -\frac{1}{2} \frac{\partial B_{z0}}{\partial z} \frac{\partial}{\partial \mathbf{x}_\perp} \cdot \mathbf{x}_\perp + \frac{\partial}{\partial z} B_{z0} = 0$$

but not curl free within the vacuum aperture:

$$\nabla \times \mathbf{B}^a = \frac{1}{2} \frac{\partial^2 B_{z0}(z)}{\partial z^2} (-\hat{\mathbf{x}} y + \hat{\mathbf{y}} x)$$
$$= \frac{1}{2} \frac{\partial^2 B_{z0}(z)}{\partial z^2} r(-\hat{\mathbf{x}} \sin\theta + \hat{\mathbf{y}} \cos\theta) = \frac{1}{2} \frac{\partial^2 B_{z0}(z)}{\partial z^2} r\hat{\theta}$$

Nonlinear terms needed to satisfy 3D Maxwell equations

Solenoid equations of motion:

Insert field components into equations of motion and collect terms

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' - \frac{B'_{z0}(s)}{2[B\rho]} y - \frac{B_{z0}(s)}{[B\rho]} y' = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

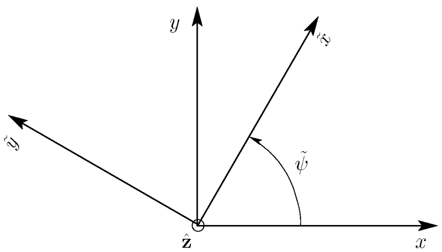
$$y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} y' + \frac{B'_{z0}(s)}{2[B\rho]} x + \frac{B_{z0}(s)}{[B\rho]} x' = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y}$$

$$[B\rho] \equiv \frac{\gamma_b \beta_b m c}{q} = \text{Rigidity} \qquad \frac{B_{z0}(s)}{[B\rho]} = \frac{\omega_c(s)}{\gamma_b \beta_b c}$$

$$\omega_c(s) = \frac{q B_{z0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}$$

Equations are linearly cross-coupled in the applied field terms
- x equation depends on y, y'
- y equation depends on x, x'

It can be shown (see: Appendix B) that the linear cross-coupling in the applied field can be removed by an s-varying transformation to a rotating "Larmor" frame:



$\tilde{\ }$ used to denote rotating frame variables

$$\tilde{x} = \quad x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\; k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_b c}$$
= Larmor wave number

$s = s_i$ defines initial condition

If the beam space-charge is *axisymmetric*:

$$\frac{\partial\phi}{\partial\mathbf{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\mathbf{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\mathbf{x}_\perp}{r}$$

then the space-charge term also decouples under the Larmor transformation and the equations of motion can be expressed in fully uncoupled form:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$
$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

$$\kappa(s) = k_L^2(s) \equiv \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2$$

Will demonstrate this in problems for the simple case of:

$$B_{z0}(s) = \text{const}$$

Because Larmor frame equations are in the same form as continuous and quadrupole focusing with a different κ, for solenoidal focusing we implicitly work in the Larmor frame and simplify notation by dropping the tildes:

$$\tilde{\mathbf{x}}_\perp \rightarrow \mathbf{x}_\perp$$

/// Aside: Notation:

A common theme of this class will be to introduce new effects and generalizations while keeping formulations looking as similar as possible to the the most simple representations given. When doing so, we will often use "tildes" to denote transformed variables to stress that the new coordinates have, in fact, a more complicated form that must be interpreted in the context of the analysis being carried out. Some examples:

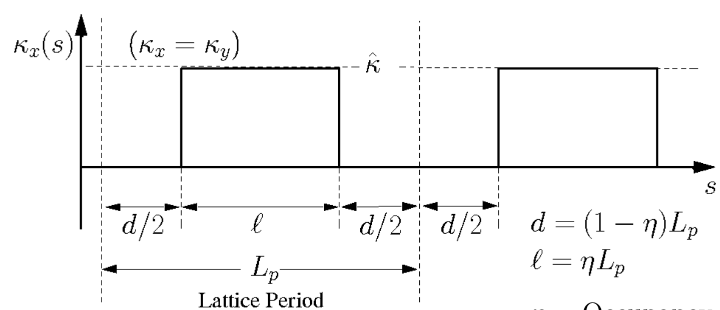
- Larmor frame transformations for Solenoidal focusing
  - See: Appendix B
- Normalized variables for analysis of accelerating systems
  - See: S10
- Coordinates expressed relative to the beam centroid
  - See: S.M. Lund, lectures on Transverse Centroid and Envelope Model
- Variables used to analyze Einzel lenses
  - See: J.J. Barnard, Introductory Lectures

///

Solenoid periodic lattices can be formed similarly to the quadrupole case

- Drifts placed between solenoids of finite axial length
  - Allows space for diagnostics, pumping, acceleration cells, etc.
- Analogous equivalence cases to quadrupole
  - Piecewise constant κ often used
- Fringe can be more important for solenoids

Simple hard-edge solenoid lattice with piecewise constant κ

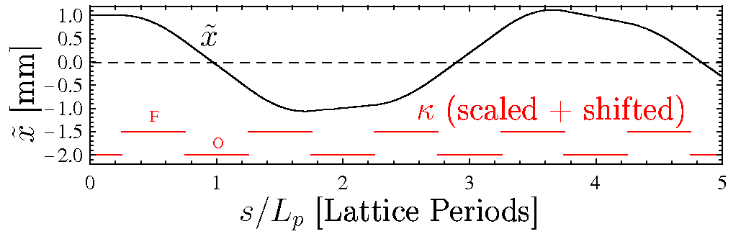


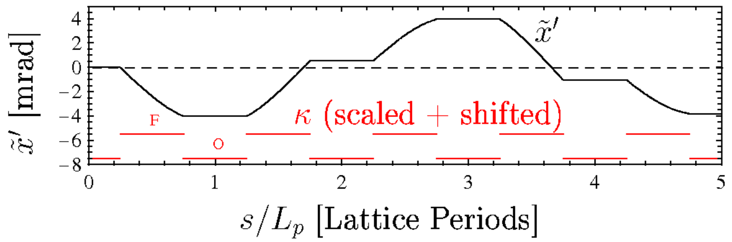
$$d = (1-\eta)L_p$$
$$\ell = \eta L_p$$
$$\eta = \text{Occupancy} \in (0, 1]$$

/// Example: Larmor Frame Particle Orbits in a Periodic Solenoidal Focusing Lattice: $\tilde{x}$ – $\tilde{x}'$ phase-space for hard edge elements and applied fields

$L_p = 0.5$ m $\kappa = 20$ rad/m^2 in Solenoids $\tilde{x}(0) = 1$ mm

$\eta = 0.5$ $\phi \simeq 0$ $\gamma_b\beta_b =$ const $\tilde{x}'(0) = 0$





///

Comments on Orbits:

See Appendix C for details on calculation

- Discontinuous fringe of hard-edge model must be treated carefully if integrating in the laboratory-frame.

Larmor-frame orbits strongly deviate from simple harmonic form due to periodic focusing

- Multiple harmonics present
- Less complicated that quadrupole AG focusing case when interpreted in the Larmor frame due to the optic being focusing in both planes

Orbits can be transformed back into the Laboratory frame using Larmor transform (see: Appendix B and Appendix C)

- Laboratory frame orbit exhibits more complicated *x-y* plane coupled oscillatory structure

Will find later that if the focusing is sufficiently strong, the orbit can become unstable (see: S5)

y-orbits have same properties as the *x*-orbits due to the equations being decoupled and identical in form in each plane

Some properties of particle orbits in solenoids with $\kappa =$ const will be analyzed in the problem sets

///

S2F: Summary of Transverse Particle Equations of Motion

In linear applied focusing channels, without momentum spread or radiation, the particle equations of motion in both the *x*- and *y*-planes expressed as:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi$$

$$\kappa_x(s) = x\text{-focusing function of lattice}$$

$$\kappa_y(s) = y\text{-focusing function of lattice}$$

Common focusing functions:

Continuous:
$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Quadrupole (Electric or Magnetic):
$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

Solenoidal (equations must be interpreted in Larmor Frame: see Appendix B):
$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:
$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide (see later material/lecture)
BUT not physically realizable (see S2B)

Quadrupole:
$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x}, & \text{Magnetic} \end{cases}$$

Solenoid:
$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{mc}$$

It is instructive to review the structure of solutions of the transverse particle equations of motion in the absence of:

Space-charge: $\dfrac{\partial \phi}{\partial x} \sim \dfrac{\partial \phi}{\partial y} \sim 0$

Acceleration: $\gamma_b \beta_b \simeq \text{const} \qquad \Longrightarrow \dfrac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} \simeq 0$

In this simple limit, the *x* and *y*-equations are of the same Hill's Equation form:

$$\boxed{\begin{aligned} x'' + \kappa_x(s) x &= 0 \\ y'' + \kappa_y(s) y &= 0 \end{aligned}}$$

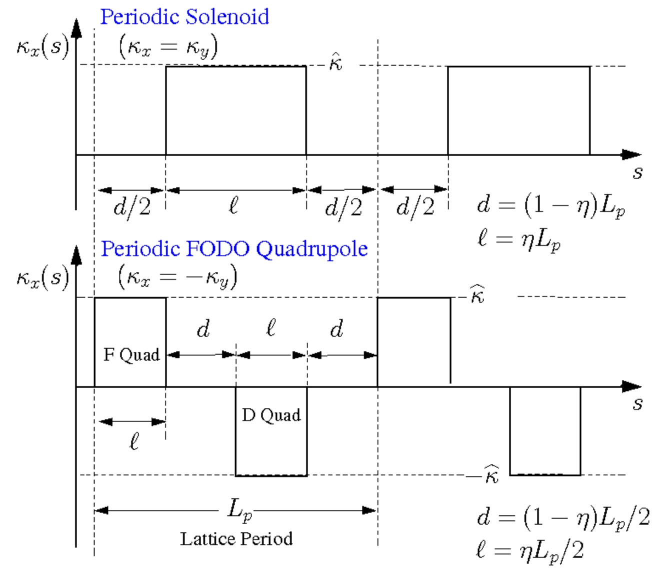
These equations are central to transverse dynamics in conventional accelerator physics (weak space-charge and acceleration)

- Will study how solutions change with space-charge in later lectures

In many cases beam transport lattices are designed where the applied focusing functions are periodic:

$$\boxed{\begin{aligned} \kappa_x(s + L_p) &= \kappa_x(s) \\ \kappa_y(s + L_p) &= \kappa_y(s) \end{aligned} \qquad L_p = \text{Lattice Period}}$$

Common, simple examples of periodic lattices:



However, the focusing functions need not be periodic:

Often take periodic or continuous in this class for simplicity of interpretation

Focusing functions can vary strongly in many common situations:

Matching and transition sections

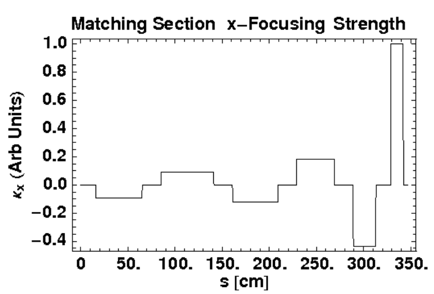
Strong acceleration

Significantly different elements can occur within periods of lattices in rings

- "Panofsky" type (wide aperture along one plane) quadrupoles for beam insertion and extraction in a ring

Example of Non-Periodic Focusing Functions: Beam Matching Section

Maintains alternating-gradient structure but not quasi-periodic



Example corresponds to High Current Experiment Matching Section (hard edge equivalent) at LBNL (2002)

Equations presented in this section apply to a single particle moving in a beam under the action of linear applied focusing forces. In the remaining sections, we will (mostly) neglect space-charge ($\phi \to 0$) as is conventional in the standard theory of low-intensity accelerators.

What we learn from treatment will later aid analysis of space-charge effects

- Appropriate variable substitutions will be made to apply results

Important to understand basic applied field dynamics since space-charge complicates

- Results in plasma-like collective response

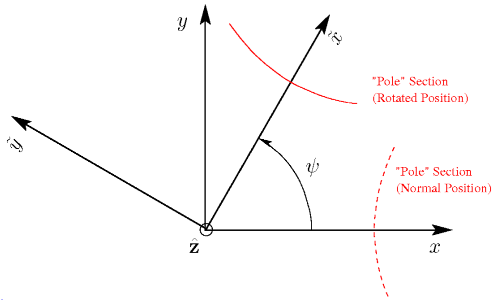
/// Example: We will see in Transverse Centroid and Envelope Descriptions of Beam Evolution that the linear particle equations of motion can be applied to analyze the evolution of a beam when image charges are neglected

$$\begin{aligned} x \to x_c &\equiv \langle x \rangle_\perp \quad x - \text{centroid} \\ y \to y_c &\equiv \langle y \rangle_\perp \quad y - \text{centroid} \end{aligned}$$

///

Appendix A: Quadrupole Skew Coupling

Consider a quadrupole actively rotated through an angle ψ about the *z*-axis:



Transforms

$$\begin{aligned}\tilde{x} &= x\cos\psi + y\sin\psi \\ \tilde{y} &= -x\sin\psi + y\cos\psi\end{aligned}$$

$$\begin{aligned}x &= \tilde{x}\cos\psi - \tilde{y}\sin\psi \\ y &= \tilde{x}\sin\psi + \tilde{y}\cos\psi\end{aligned}$$

Normal Orientation Fields

Electric

$$\begin{aligned}E_x^a &= -Gx \\ E_y^a &= Gy\end{aligned}$$

Magnetic

$$\begin{aligned}B_x^a &= Gy \\ B_y^a &= Gx\end{aligned}$$

$$\begin{aligned}G &= G(s) \\ &= \text{Field Gradient (Electric or Magnetic)}\end{aligned}$$

Note: units of G different in electric and magnetic cases

A1

Rotated Fields

Electric

$$\begin{aligned}E_x^a &= E_{\tilde{x}}^a\cos\psi - E_{\tilde{y}}^a\sin\psi & E_{\tilde{x}}^a &= -G\tilde{x} = -G(x\cos\psi + y\sin\psi) \\ E_y^a &= E_{\tilde{x}}^a\sin\psi + E_{\tilde{y}}^a\cos\psi & E_{\tilde{y}}^a &= G\tilde{y} = G(-x\sin\psi + y\cos\psi)\end{aligned}$$

Combine equations, collect terms, and apply trigonometric identities to obtain:

$$\boxed{\begin{aligned}E_x^a &= -G\cos(2\psi)x - G\sin(2\psi)y \\ E_y^a &= -G\sin(2\psi)x + G\cos(2\psi)y\end{aligned}}$$

$$\begin{aligned}2\sin\psi\cos\psi &= \sin(2\psi) \\ \cos^2\psi - \sin^2\psi &= \cos(2\psi)\end{aligned}$$

Magnetic

$$\begin{aligned}B_x^a &= B_{\tilde{x}}^a\cos\psi - B_{\tilde{y}}^a\sin\psi & B_{\tilde{x}}^a &= G\tilde{y} = G(-x\sin\psi + y\cos\psi) \\ B_y^a &= B_{\tilde{x}}^a\sin\psi + B_{\tilde{y}}^a\cos\psi & B_{\tilde{y}}^a &= G\tilde{x} = G(x\cos\psi + y\sin\psi)\end{aligned}$$

Combine equations, collect terms, and apply trigonometric identities to obtain:

$$\boxed{\begin{aligned}B_x^a &= -G\sin(2\psi)x + G\cos(2\psi)y \\ B_y^a &= G\cos(2\psi)x + G\sin(2\psi)y\end{aligned}}$$

A2

For *both* electric and magnetic focusing quadrupoles, these field component projections can be inserted in the linear field Eqns of motion to obtain:

Skew Coupled Quadrupole Equations of Motion

$$\boxed{\begin{aligned}x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa\cos(2\psi)x + \kappa\sin(2\psi)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa\cos(2\psi)y + \kappa\sin(2\psi)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \\ \kappa = \begin{cases}\frac{G}{\beta_b c[B\rho]}, & \text{Electric Focusing} \\ \frac{G}{[B\rho]}, & \text{Magnetic Focusing}\end{cases}\end{aligned}}$$

System is skew coupled:
- *x*-equation depends on *y*, *y'* and *y*-equation on *x*, *x'* for $\psi \neq 0, \pi, 2\pi, \cdots$

Skew-coupling considerably complicates dynamics
- Unless otherwise specified, we consider only quadrupoles with "normal" orientation with $\psi = 0$
- Skew coupling errors or intentional skew couplings can be important
  - Leads to transfer of oscillations energy between *x* and *y*-planes
  - Invariants much more complicated to construct/interpret

A3

The skew coupled equations of motion can be alternatively derived by actively rotating the quadrupole equation of motion in the form:

$$\begin{aligned}x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' - \kappa(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}\end{aligned}$$

Steps are then identical whether quadrupoles are electric *or* magnetic

A4

Appendix B: The Larmor Transform to Express Solenoidal Focused Particle Equations of Motion in Uncoupled Form

Solenoid equations of motion:

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$
$$B_{z0}(s) = B_z^a(r=0, z=s) = \text{On-Axis Field}$$
$$[B\rho] = \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity}$$

To simplify algebra, introduce the complex coordinate

$$\underline{z} \equiv x + iy \qquad i \equiv \sqrt{-1}$$

Note* context clarifies use of i (particle index, initial cond, complex i)

Then the two equations can be expressed as a single complex equation

$$\underline{z}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\underline{z}' + i\frac{B'_{z0}(s)}{2[B\rho]}\underline{z} + i\frac{B_{z0}(s)}{[B\rho]}\underline{z}' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right)$$

B1

If the potential is also axisymmetric with $\phi = \phi(r)$:

$$\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial r}\frac{\underline{z}}{r} \qquad r \equiv \sqrt{x^2+y^2}$$

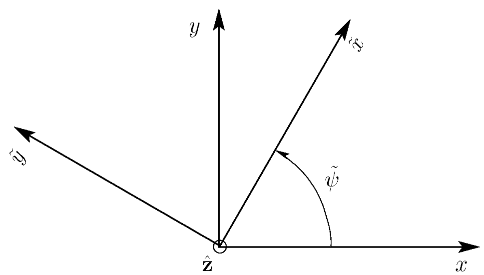
then the complex form equation of motion reduces to:

$$\underline{z}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\underline{z}' + i\frac{B'_{z0}(s)}{2[B\rho]}\underline{z} + i\frac{B_{z0}(s)}{[B\rho]}\underline{z}' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\underline{z}}{r}$$

Following Wiedemann, Vol II, pg 82, introduce a transformed complex variable that is a local (s-varying) rotation:

$$\tilde{\underline{z}} \equiv \underline{z}e^{-i\tilde{\psi}(s)} = \tilde{x} + i\tilde{y}$$

$\tilde{\psi}(s) =$ phase-function (real-valued)



B2

Then: $\underline{z} = \tilde{\underline{z}}e^{i\tilde{\psi}}$

$$\underline{z}' = \left(\tilde{\underline{z}}' + i\tilde{\psi}'\tilde{\underline{z}}\right)e^{i\tilde{\psi}}$$
$$\underline{z}'' = \left(\tilde{\underline{z}}'' + 2i\tilde{\psi}'\tilde{\underline{z}}' + i\tilde{\psi}''\tilde{\underline{z}} - \tilde{\psi}'^2\tilde{\underline{z}}\right)e^{i\tilde{\psi}}$$

and the complex form equations of motion become:

$$\tilde{\underline{z}}'' + \left[i\left(2\tilde{\psi}' + \frac{B_{z0}}{[B\rho]}\right) + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\right]\tilde{\underline{z}}'$$
$$+ \left[-\tilde{\psi}'^2 - \frac{B_{z0}}{[B\rho]}\tilde{\psi}' + i\left(\tilde{\psi}'' + \frac{B'_{z0}}{2[B\rho]} + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{\psi}'\right)\right]\tilde{\underline{z}}$$
$$= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{\underline{z}}}{r}$$

Free to choose the form of $\tilde{\psi}$ Can choose to eliminate imaginary terms in [....] by taking:

$$\tilde{\psi}' \equiv -\frac{B_{z0}}{2[B\rho]} \qquad \Longrightarrow \qquad \tilde{\psi}'' = -\frac{B'_{z0}}{2[B\rho]} + \frac{B_{z0}}{2[B\rho]}\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}$$

B3

B4

Using these results, the complex form equations of motion reduce to:

$$\tilde{\underline{z}}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{\underline{z}}' + \left(\frac{B_{z0}}{2[B\rho]}\right)^2\tilde{\underline{z}} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{\underline{z}}}{r}$$

Or using $\tilde{\underline{z}} = \tilde{x} + i\tilde{y}$, the equations can be expressed in decoupled $\tilde{x},\ \tilde{y}$ variables in the Larmor Frame as:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$
$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$
$$\kappa_s(s) \equiv k_L^2(s) \qquad k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_bc} \qquad [B\rho] = \frac{\gamma_b\beta_b mc}{q}$$
$$= \text{Larmor Wave-Number}$$

Equations of motion are uncoupled but must be interpreted in the rotating Larmor frame

Same form as quadrupoles but with focusing function same sign in each plane

The rotational transformation to the Larmor Frame can be effected by integrating the equation for $\tilde{\psi}' = -\frac{B_{z0}}{2[B\rho]}$

$$\boxed{\tilde{\psi}(s) = -\int_{s_i}^{s} d\tilde{s}\, \frac{B_{z0}(\tilde{s})}{2[B\rho]} = -\int_{s_i}^{s} d\tilde{s}\; k_L(\tilde{s})}$$

Here, s_i is some value of *s* where the initial conditions are taken.

Take $s = s_i$ where axial field is zero for simplest interpretation (see: pg B6)

Because

$$\boxed{\tilde{\psi}' = -\frac{B_{z0}}{2[B\rho]} = \frac{\omega_c}{2\gamma_b\beta_b c}}$$

the local $\tilde{x} - \tilde{y}$ Larmor frame is rotating at ½ of the local *s*-varying cyclotron frequency

If $B_{z0} = \text{const}$, then the Larmor frame is uniformly rotating as is well known from elementary textbooks (see problem sets)

B5

The complex form phase-space transformation and inverse transformations are:

$$\underline{z} = \underline{\tilde{z}} e^{i\tilde{\psi}} \qquad \underline{\tilde{z}} = \underline{z} e^{-i\tilde{\psi}}$$

$$\underline{z}' = \left(\underline{\tilde{z}}' + i\tilde{\psi}'\underline{\tilde{z}}\right) e^{i\tilde{\psi}} \qquad \underline{\tilde{z}}' = \left(\underline{z}' - i\tilde{\psi}'\underline{z}\right) e^{-i\tilde{\psi}}$$

$$\underline{z} = x + iy \qquad \underline{\tilde{z}} = \tilde{x} + i\tilde{y} \qquad \tilde{\psi}' = -k_L$$

$$\underline{z}' = x' + iy' \qquad \underline{\tilde{z}}' = \tilde{x}' + i\tilde{y}'$$

Apply to:

Project initial conditions from lab-frame when integrating equations

Project integrated solution back to lab-frame to interpret solution

If the initial condition $s = s_i$ is taken outside of the magnetic field where $B_{z0}(s_i) = 0$, then:

$$\tilde{x}(s = s_i) = x(s = s_i) \qquad \tilde{x}'(s = s_i) = x'(s = s_i)$$

$$\tilde{y}(s = s_i) = y(s = s_i) \qquad \tilde{y}'(s = s_i) = y'(s = s_i)$$

$$\underline{\tilde{z}}(s = s_i) = \underline{z}(s = s_i) \qquad \underline{\tilde{z}}'(s = s_i) = \underline{z}'(s = s_i)$$

B6

The transform and inverse transform between the laboratory and rotating frames can then be applied to project initial conditions into the rotating frame for integration and then the rotating frame solution back into the laboratory frame.

Using the real and imaginary parts of the complex-valued transformations:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \tilde{\mathbf{M}}_r(s|s_i) \cdot \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix} \qquad \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix} = \tilde{\mathbf{M}}_r^{-1}(s|s_i) \cdot \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

$$\tilde{\mathbf{M}}_r(s|s_i) = \begin{pmatrix} \cos\tilde{\psi} & 0 & -\sin\tilde{\psi} & 0 \\ k_L \sin\tilde{\psi} & \cos\tilde{\psi} & k_L\cos\tilde{\psi} & -\sin\tilde{\psi} \\ \sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ -k_L\cos\tilde{\psi} & \sin\tilde{\psi} & k_L\sin\tilde{\psi} & \cos\tilde{\psi} \end{pmatrix}$$

$$\tilde{\mathbf{M}}_r^{-1}(s|s_i) = \begin{pmatrix} \cos\tilde{\psi} & 0 & \sin\tilde{\psi} & 0 \\ k_L \sin\tilde{\psi} & \cos\tilde{\psi} & -k_L\cos\tilde{\psi} & \sin\tilde{\psi} \\ -\sin\tilde{\psi} & 0 & \cos\tilde{\psi} & 0 \\ k_L\cos\tilde{\psi} & -\sin\tilde{\psi} & k_L\sin\tilde{\psi} & \cos\tilde{\psi} \end{pmatrix}$$

Here we used:

$$\tilde{\psi}' = -k_L$$

and it can be verified that:

$$\tilde{\mathbf{M}}_r^{-1} = \text{Inverse}[\tilde{\mathbf{M}}_r]$$

B7

Appendix C: Transfer Matrices for Hard-Edge Solenoidal Focusing

Using results and notation from Appendix B, derive transfer matrix for single particle orbit with:

No space-charge

No momentum spread

First, the solution to the Larmor-frame equations of motion:

$$\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_L(s)\tilde{x} = 0$$

$$\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa_L(s)\tilde{y} = 0$$

Can be expressed as:

$$\boxed{\begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix}_s = \tilde{\mathbf{M}}_L(s|s_i) \cdot \begin{pmatrix} \tilde{x} \\ \tilde{x}' \\ \tilde{y} \\ \tilde{y}' \end{pmatrix}_{s=s_i}}$$

C1

Transforming the solution back to the laboratory frame:

From project of initial conditions to Larmor Frame

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_s = \tilde{\mathbf{M}}_r(s|s_i)\cdot\tilde{\mathbf{M}}_L(s|s_i)\cdot\underbrace{\tilde{\mathbf{M}}_r^{-1}(s_i|s_i)}_{= I \text{ Identity Matrix}}\cdot\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s=s_i}$$

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_s \equiv \mathbf{M}(s|s_i)\cdot\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s=s_i} = \tilde{\mathbf{M}}_r(s|s_i)\cdot\tilde{\mathbf{M}}_L(s|s_i)\cdot\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_{s=s_i}$$

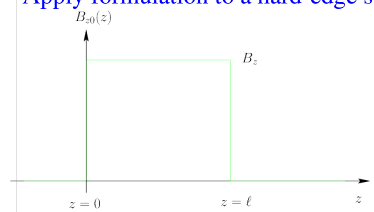
$$\mathbf{M}(s|s_i) = \tilde{\mathbf{M}}_r(s|s_i)\cdot\tilde{\mathbf{M}}_L(s|s_i)$$

Care must be taken when applying to discontinuous (hard-edge) field models of solenoids to correctly calculate transfer matrices

- Fringe field influences beam "spin-up" and "spin-down" entering and exiting the magnet

C2

Apply formulation to a hard-edge solenoid with no acceleration [$(\gamma_b\beta_b)' = 0$]:



$$B_{z0}(s) = B_z\,[\Theta(z) - \Theta(z-\ell)]$$
$$B_z = \text{const} = \text{Hard-Edge Field}$$
$$\ell = \text{const} = \text{Hard-Edge Magnet Length}$$

Note coordinate choice: z=0 is start of magnet

Calculate the Larmor-frame transfer matrix in $0 \le z \le \ell$:

$$\tilde{x}'' + \kappa_L\tilde{x} = 0 \qquad k_L = \frac{qB_z}{2\gamma_b\beta_b mc} = \frac{B_z}{2[B\rho]} = \text{const}$$
$$\tilde{y}'' + \kappa_L\tilde{y} = 0$$

$$\tilde{\mathbf{M}}_L(s|0) = \begin{pmatrix} C & S/k_L & 0 & 0 \\ -k_L S & C & 0 & 0 \\ 0 & 0 & C & S/k_L \\ 0 & 0 & -k_L S & C \end{pmatrix}$$

$$C \equiv \cos(k_L z)$$
$$S \equiv \sin(k_L z)$$

C3

From this we obtain the rotation matrix within the magnet $0 < z < \ell$:

$$\tilde{\mathbf{M}}_r(s|0) = \begin{pmatrix} C & 0 & S & 0 \\ -k_L S & C & k_L C & S \\ -S & 0 & C & 0 \\ -k_L C & -S & -k_L S & C \end{pmatrix}$$

With special magnet end-forms (simply evaluate $\tilde{\mathbf{M}}_r$ at ends):

$$\tilde{\mathbf{M}}_r(0^+|0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{\mathbf{M}}_r(\ell+|0) = \begin{pmatrix} \cos\Phi & 0 & \sin\Phi & 0 \\ 0 & \cos\Phi & 0 & \sin\Phi \\ -\sin\Phi & 0 & \cos\Phi & 0 \\ 0 & -\sin\Phi & 0 & \cos\Phi \end{pmatrix} \qquad \Phi \equiv k_L\ell$$

C4

The lab-frame advance matrices are then (after expanding matrix products):

$0^+ \le z \le \ell^-$

$$\mathbf{M}(s|0) = \tilde{\mathbf{M}}_r(s|0)\tilde{\mathbf{M}}_L(s|0)$$
$$= \begin{pmatrix} \cos^2\phi & \frac{1}{2k_L}\sin(2\phi) & \frac{1}{2}\sin(2\phi) & \frac{1}{k_L}\sin^2\phi \\ -k_L\sin(2\phi) & \cos(2\phi) & k_L\cos(2\phi) & \sin(2\phi) \\ -\frac{1}{2}\sin(2\phi) & -\frac{1}{k_L}\sin^2\phi & \cos^2\phi & \frac{1}{2k_L}\sin(2\phi) \\ -k_L\cos(2\phi) & -\sin(2\phi) & -k_L\sin(2\phi) & \cos(2\phi) \end{pmatrix}$$
$$\phi \equiv k_L z$$

$z = \ell^+$

$$\mathbf{M}(\ell^+|0) = \tilde{\mathbf{M}}_r(\ell^+|0)\tilde{\mathbf{M}}_L(\ell^+|0)$$
$$= \begin{pmatrix} \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) & \frac{1}{2}\sin(2\Phi) & \frac{1}{k_L}\sin^2\Phi \\ -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi & -k_L\sin^2\Phi & \frac{1}{2}\sin(2\Phi) \\ -\frac{1}{2}\sin(2\Phi) & -\frac{1}{k_L}\sin^2\Phi & \cos^2\Phi & \frac{1}{2k_L}\sin(2\Phi) \\ k_L\sin^2\Phi & -\frac{1}{2}\sin(2\Phi) & -\frac{k_L}{2}\sin(2\Phi) & \cos^2\Phi \end{pmatrix}$$
$$\Phi \equiv k_L\ell$$

Note that due to discontinuous fringe field:

$$\mathbf{M}(0^+|0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_L & 0 \\ 0 & 0 & 1 & 0 \\ -k_L & 0 & 0 & 1 \end{pmatrix} \neq I$$

Fringe going in kicks angles of beam

C5

$$\mathbf{M}(\ell^-|0) \neq \mathbf{M}(\ell^+|0)$$

Due to fringe exiting kicking angles of beam

In more realistic model with a continuously varying fringe to zero, all transfer matrix components will vary continuously across boundaries

- Still important to get this right in idealized designs often taken as a first step!

Focusing kicks on particles entering/exiting the solenoid can be calculated as:

Entering:

$$\boxed{\begin{aligned} x(0^+) &= x(0^-) & x'(0^+) &= x'(0^-) + k_L y(0^-) \\ y(0^+) &= y(0^-) & y'(0^+) &= y'(0^-) - k_L x(0^-) \end{aligned}}$$

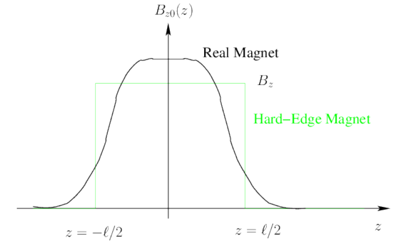
Exiting:

$$\boxed{\begin{aligned} x(\ell^+) &= x(\ell^-) & x'(\ell^+) &= x'(\ell^-) - k_L y(\ell^-) \\ y(\ell^+) &= y(\ell^-) & y'(\ell^+) &= y'(\ell^-) + k_L x(\ell^-) \end{aligned}}$$

Beam spins up/down on entering exiting the (abrupt) magnetic fringe field
Sense of rotation changes with entry/exit of hard-edge field.

C6

The transfer matrix for the hard-edge solenoid is exact within the context of linear optics. However, real solenoid magnets have an axial fringe field. An obvious need is how to best set the hard-edge parameters B_z, ℓ from the real fringe field.



Hard-Edge and Real Magnets axially centered to compare

Simple physical motivated prescription by requiring:

1) Equivalent Linear Focus Impulse $\propto \int dz\, k_L^2 \propto \int dz B_{z0}^2$

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} dz\; B_{z0}^2(z) \;= \ell B_z^2}$$

2) Equivalent Net Larmor Rotation Angle $\propto \int dz\; k_L \propto \int dz\; B_{z0}$

$$\Longrightarrow \boxed{\int_{-\infty}^{\infty} dz\; B_{z0}(z) \;= \ell B_z}$$

C7

Solve 1) and 2)

$$\boxed{\begin{aligned} B_z &= \frac{\int_{-\infty}^{\infty} dz\; B_{z0}^2(z)}{\int_{-\infty}^{\infty} dz\; B_{z0}(z)} \\ \ell &= \frac{\left[\int_{-\infty}^{\infty} dz\; B_{z0}(z)\right]^2}{\int_{-\infty}^{\infty} dz\; B_{z0}^2(z)} \end{aligned}}$$

Numerical tests show close correspondence between the actual and hard-edge linear optics model when this correspondence is applied to "typical" solenoids in lab use.

C8

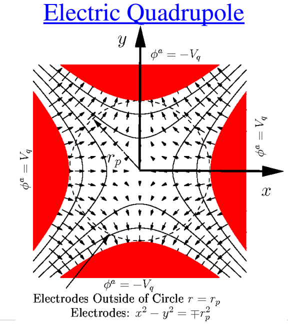
S3: Description of Applied Focusing Fields
S3A: Overview

Applied fields for focusing, bending, and acceleration enter the equations of motion via:

$$\mathbf{E}^a = \text{Applied Electric Field}$$
$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

Generally, these fields are produced by sources (often static or slowly varying in time) located outside an aperture or so-called pipe radius $r = r_p$. For example, the electric and magnetic quadrupoles of S2:

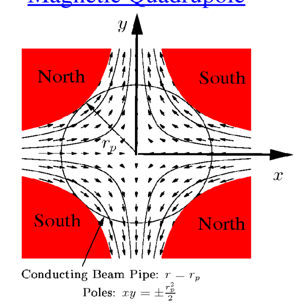
Electric Quadrupole



Hyperbolic material surfaces outside pipe radius

$r = r_p$

Magnetic Quadrupole



The fields of such classes of magnets obey the vacuum Maxwell Equations within the aperture:

$$\nabla\cdot\mathbf{E}^a = 0 \qquad \nabla\cdot\mathbf{B}^a = 0$$
$$\nabla\times\mathbf{E}^a = -\frac{\partial}{\partial t}\mathbf{B}^a \qquad \nabla\times\mathbf{B}^a = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a$$

If the fields are static or sufficiently slowly varying (quasistatic) where the time derivative terms can be neglected, then the fields in the aperture will obey the static vacuum Maxwell equations:

$$\nabla\cdot\mathbf{E}^a = 0 \qquad \nabla\cdot\mathbf{B}^a = 0$$
$$\nabla\times\mathbf{E}^a = 0 \qquad \nabla\times\mathbf{B}^a = 0$$

In general, optical elements are tuned to limit the strength of nonlinear field terms so the beam experiences primarily linear applied fields.

- Linear fields allow better preservation of beam quality

Removal of *all* nonlinear fields cannot be accomplished

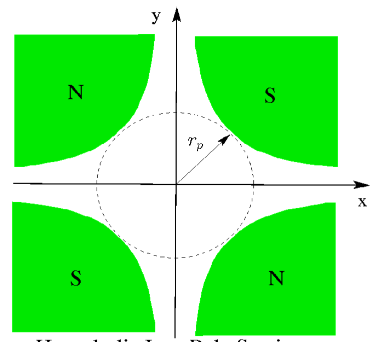
- 3D structure of the Maxwell equations precludes for finite geometry optics
- Even in finite geometries deviations from optimal structures and symmetry will result in nonlinear fields

As an example of this, when an ideal 2D iron magnet with infinite hyperbolic poles is truncated radially for finite 2D geometry, this leads to nonlinear focusing fields even in 2D:

- Truncation necessary along with confinement of return flux in yoke

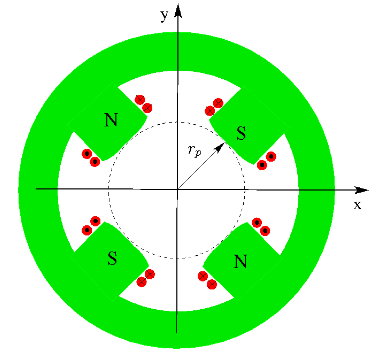
Cross-Sections of Iron Quadrupole Magnets

Ideal (infinite geometry)



Hyperbolic Iron Pole Sections (infinite)

Practical (finite geometry)



Shaped Iron Pole Sections (finite)

The design of optimized electric and magnetic optics for accelerators is a specialized topic with a vast literature. It is not be possible to cover this topic in this brief survey. In the remaining part of this section we will overview a limited subset of material on magnetic optics including:

- (see: S3B) Magnetic field expansions for focusing and bending
- (see: S3C) Hard edge equivalent models
- (see: S3D) 2D multipole models and nonlinear field scalings
- (see: S3E) Good field radius

Much of the material presented can be immediately applied to static Electric Optics since the vacuum Maxwell equations are the same for static Electric $\mathbf{E}^a$ and Magnetic $\mathbf{B}^a$ fields in vacuum.

S3B: Magnetic Field Expansions for Focusing and Bending

Forces from transverse $(B_z^a = 0)$ magnetic fields enter the transverse equations of motion (see: S1, S2) via:

Force: $\mathbf{F}_\perp^a \simeq q\beta_b c\hat{\mathbf{z}}\times\mathbf{B}_\perp^a$

Field: $\mathbf{B}_\perp^a = \hat{\mathbf{x}}B_x^a + \hat{\mathbf{y}}B_y^a$

Combined these give:

$$F_x^a \simeq -q\beta_b c B_y^a$$
$$F_y^a \simeq q\beta_b c B_x^a$$

Field components entering these expressions can be expanded about $\mathbf{x}_\perp = 0$

- Element center and design orbit taken to be at $\mathbf{x}_\perp = 0$

$$B_x^a = \overset{1}{B_x^a(0)} + \overset{2}{\frac{\partial B_x^a}{\partial y}(0)y} + \overset{3}{\frac{\partial B_x^a}{\partial x}(0)x}$$
$$+\frac{1}{2}\frac{\partial^2 B_x^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_x^a}{\partial x\partial y}(0)xy + \frac{1}{2}\frac{\partial B_x^a}{\partial y^2}(0)y^2 + \cdots$$

Nonlinear Focus

$$B_y^a = \overset{1}{B_y^a(0)} + \overset{2}{\frac{\partial B_y^a}{\partial x}(0)x} + \overset{3}{\frac{\partial B_y^a}{\partial y}(0)y}$$
$$+\frac{1}{2}\frac{\partial^2 B_y^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_y^a}{\partial x\partial y}(0)xy + \frac{1}{2}\frac{\partial B_y^a}{\partial y^2}(0)y^2 + \cdots$$

Nonlinear Focus

Terms:
1: Dipole Bend
2: Normal Quad Focus
3: Skew Quad Focus

Sources of undesired nonlinear applied field components include:

- Intrinsic finite 3D geometry and the structure of the Maxwell equations
- Systematic errors or sub-optimal geometry associated with practical trade-offs in fabricating the optic
- Random construction errors in individual optical elements
- Alignment errors of magnets in the lattice giving field projections in unwanted directions
- Excitation errors effecting the field strength
  - Currents in coils not correct and/or unbalanced

More advanced treatments exploit less simple power-series expansions to express symmetries more clearly:

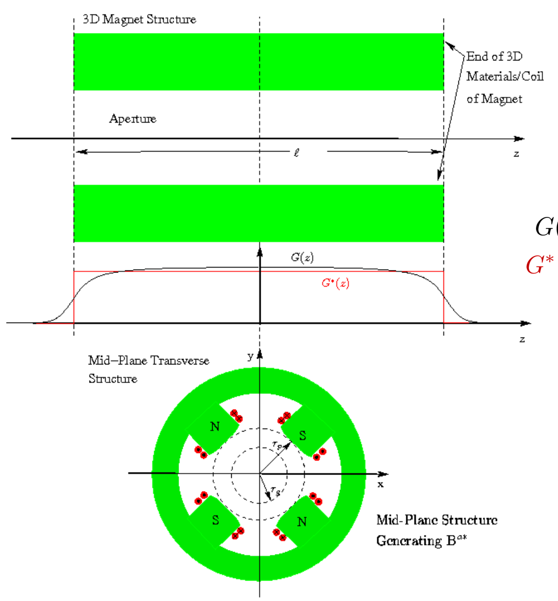
- Maxwell equations constrain structure of solutions
  - Expansion coefficients are NOT all independent
- Forms appropriate for bent coordinate systems in dipole bends can become complicated

S3C: Hard Edge Equivalent Models

Real 3D magnets can often be modeled with sufficient accuracy by 2D hard-edge "equivalent" magnets that give the same approximate focusing impulse to the particle as the full 3D magnet

- Objective is to provide same approximate applied focusing "kick" to particles with different gradient focusing gradient functions $G(s)$

See Figure Next Slide



$G(z) = \text{3D Field Gradient}$

$G^*(z) = \text{Hard-Edge Equivalent Field Gradient}$

Many prescriptions exist for calculating the effective axial length and strength of hard-edge equivalent models

- See Review: Lund and Bukh, PRSTAB 7 204801 (2004), Appendix C

Here we overview a simple equivalence method that has been shown to work well:

For a relatively long, but finite axial length magnet with 3D gradient function:

$$G(z) \equiv \left.\frac{\partial B_x^a}{\partial y}\right|_{x=y=0}$$

Take hard-edge equivalent parameters:

- Assume $z = 0$ at the axial magnet mid-plane

Gradient: $$G^* \equiv G(z=0)$$

Axial Length: $$\ell \equiv \frac{1}{G(z=0)} \int_{-\infty}^{\infty} dz\, G(z)$$

- More advanced equivalences can be made based more on particle optics
  - Disadvantage of such methods is "equivalence" changes with particle energy and must be revisited as optics are tuned

S3D: 2D Transverse Multipole Magnetic Fields

In many cases, it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$\overline{B_x}(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_x^a(x,y,z)$$

$$\overline{B_y}(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_y^a(x,y,z)$$

2D Effective Fields   3D Fields

Operating on the vacuum Maxwell equations with: $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{\ell}\cdots$

yields the (exact) 2D Transverse Maxwell equations :

$$\frac{\partial\overline{B_x}(x,y)}{\partial y} = \frac{\partial\overline{B_y}(x,y)}{\partial x} \quad \Leftarrow \text{ From } \quad \nabla\times\mathbf{B}=0$$

$$\frac{\partial\overline{B_x}(x,y)}{\partial x} = -\frac{\partial\overline{B_y}(x,y)}{\partial y} \quad \Leftarrow \text{ From } \quad \nabla\cdot\mathbf{B}=0$$

These equations are recognized as the Cauchy-Riemann conditions for a complex field variable:

$$\underline{B}^* \equiv \overline{B_x} - i\overline{B_y} \qquad i\equiv\sqrt{-1}$$

to be an analytical function of the complex variable:

$$\underline{z} \equiv x + iy \qquad i\equiv\sqrt{-1}$$

Notation:
Underlines denote complex variables where confusion may arise

Note the complex field which is an analytic function of $\underline{z} = x+iy$ is $\underline{B}^* = \overline{B_x} - i\overline{B_y}$ NOT $\underline{B} = \overline{B_x} + i\overline{B_y}$. This is *not* a typo and is necessary for $\underline{B}^*$ to satisfy the Cauchy-Riemann conditions.

See problem sets for illustration

It follows that $\underline{B}^*(\underline{z})$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $\underline{B}^*(\underline{z})$ as a Laurent Series within the vacuum aperture as:

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty}\underline{b}_n\underline{z}^{n-1}$$

$$\underline{b}_n = \text{const (complex)}$$

$$n = \text{Multipole Index}$$

The $\underline{b}_n$ are called "multipole coefficients" and give the structure of the field. The multipole coefficients can be resolved into real and imaginary parts as:

$$\underline{b}_n = \mathcal{A}_n - i\mathcal{B}_n$$

$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$

$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Some algebra identifies the polynomial symmetries of low-order terms as:

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}$

Index n	Name	Normal ($\mathcal{A}_n = 0$) $\overline{B_x}/\mathcal{B}_n$	$\overline{B_y}/\mathcal{B}_n$	Skew ($\mathcal{B}_n = 0$) $\overline{B_x}/\mathcal{A}_n$	$\overline{B_y}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	x^2-y^2	x^2-y^2	$-2xy$
4	Octupole	$3x^2y-y^3$	x^3-3xy^2	x^3-3xy^2	$-3x^2y+y^3$
5	Decapole	$4x^3y-4xy^3$	$x^4-6x^2y^2+y^4$	$x^4-6x^2y^2+y^4$	$-4x^3y+4xy^3$

Comments:

Reason for pole names most apparent from polar representation (see following pages) and sketches of the magnetic pole structure

Caution: In so-called "US notation", poles are labeled with index *n* -> *n* -1

- Arbitrary in 2D but US choice not good notation in 3D generalizations

Comments continued:

Normal and Skew symmetries can be taken as a symmetry *definition.* But this choice makes sense for *n* = 2 quadrupole focusing terms:

$$\overline{F_x^a} = -q\beta_b c\overline{B_y} \quad = -q\beta_b cB_y(\mathcal{B}_2 x - \mathcal{A}_2 y)$$

$$\overline{F_y^a} = \ q\beta_b c\overline{B_x} \quad = \ q\beta_b cB_y(\mathcal{B}_2 y + \mathcal{A}_2 x)$$

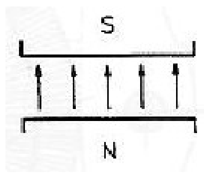
In equations of motion:

Normal $\Rightarrow$ $\mathcal{B}_2$: x-eqn, x-focus y-eqn, y-defocus

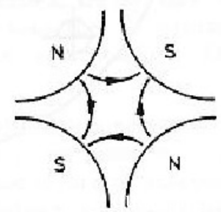
Skew $\Rightarrow$ $\mathcal{A}_2$: x-eqn, y-defocus y-eqn, x-defocus

Magnetic Pole Symmetries (normal orientation):

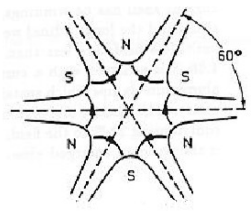
Dipole (n=1)



Quadrupole (n=2)



Sextupole (n=3)



Actively rotate normal field structures clockwise through an angle of $\pi/(2n)$ for skew field component symmetries

Multipole scale/units

Frequently, in the multipole expansion:

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}$$

the multipole coefficients $\underline{b}_n$ are rescaled as

$$\underline{b}_n \rightarrow \underline{b}_n r_p^{n-1} \qquad r_p = \text{Aperture "Pipe" Radius}$$

Closest radius of approach of magnetic sources and/or aperture materials

so that the expansions becomes

$$\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

Advantages of alternative notaiton:

- Multipoles $\underline{b}_n$ given directly in field units regardless of index *n*
- Scaling of field amplitudes with radius within the magnet bore becomes clear

Scaling of Fields produced by multipole term:

Higher order multipole coefficients (larger *n* values) leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation for $\underline{z}$, $\underline{b}_n$

$$\underline{z} = x + iy = re^{i\theta} \qquad r = \sqrt{x^2+y^2}, \quad \theta = \arctan[y,x]$$

$$\underline{b}_n = |\underline{b}_n| e^{i\psi_n} \qquad \psi_n = \text{Real Const}$$

Thus, the nth order multipole terms scale as

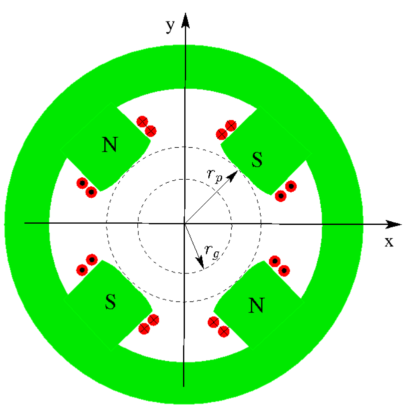
$$\underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1} = |\underline{b}_n| \left(\frac{r}{r_p}\right)^{n-1} e^{i[(n-1)\theta + \psi_n]}$$

- Unless the coefficient $|\underline{b}_n|$ is very large, high order terms in *n* will become small rapidly as r_p decreases
- Better field quality can be obtained for a given magnet design by simply making the clear bore r_p larger, or alternatively using smaller bundles (more tight focus) of particles
  - Larger bore machines/magnets cost more. So designs become trade-off between cost and performance.
  - Stronger focusing to keep beam from aperture can be unstable (see: S5)

S3E: Good Field Radius

Often a magnet design will have a so-called "good-field" radius $r = r_g$ that the maximum field errors are specified on.

- In superior designs the good field radius can be around ~70% or more of the clear bore aperture to the beginning of material structures of the magnet.
- Beam particles should evolve with radial excursions with $r < r_g$



$$r_p = \text{Clear Bore Radius} \sim \text{Pole Radius Typical}$$

$$r_g = \text{Good Field Radius} \sim 70\%\ r_p \text{ Typical}$$

Comments:

- Particle orbits are designed to remain within radius r_g
- Field error statements are readily generalized to 3D since:

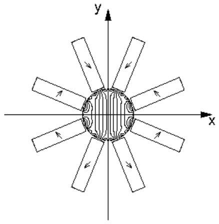
$$\nabla \cdot \mathbf{B}^a = 0, \quad \nabla \times \mathbf{B}^a = 0 \quad \Longrightarrow \quad \nabla^2 \mathbf{B}^a = 0$$

and therefore each component of $\mathbf{B}^a$ satisfies a Laplace equation within the vacuum aperture. Therefore, field errors decrease when moving within a source-free region.

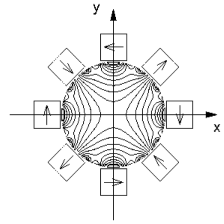
S3F: Example Permanent Magnet Assemblies

A few examples of practical permanent magnet assemblies with field contours are provided to illustrate error field structures in practical devices

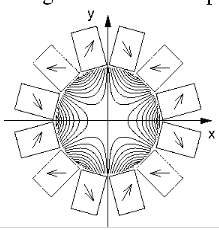
8 Rectangular Block Dipole



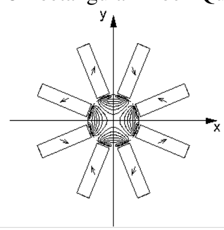
8 Square Block Quadrupole



12 Rectangular Block Sextupole



8 Rectangular Block Quadrupole



For more info on permanent magnet design see: Lund and Halbach, Fusion Engineering Design, **32-33**, 401-415 (1996)

S4: Transverse Particle Equations of Motion with Nonlinear Applied Fields S4A: Overview

In S1 we showed that the particle equations of motion can be expressed as:

$$\boxed{\mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi}$$

When momentum spread is neglected and results are interpreted in a Cartesian coordinate system (no bends). In S2, we showed that these equations can be further reduced when the applied focusing fields are linear to:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi \end{aligned}}$$

where

$$\kappa_x(s) = x\text{-focusing function of lattice}$$
$$\kappa_y(s) = y\text{-focusing function of lattice}$$

describe the linear applied focusing forces and the equations are implicitly analyzed in the rotating Larmor frame when $B_z^a \neq 0$.

Lattice designs attempt to minimize nonlinear applied fields. However, the 3D Maxwell equations show that there will *always* be some finite nonlinear applied fields for an applied focusing element with finite extent. Applied field nonlinearities also result from:

- Design idealizations
- Fabrication and material errors

The largest source of nonlinear terms will depend on the case analyzed.

Nonlinear applied fields must be added back in the idealized model when it is appropriate to analyze their effects

- Common problem to address when carrying out large-scale numerical simulations to design/analyze systems

There are two basic approaches to carry this out:

- Approach 1: Explicit 3D Formulation
- Approach 2: Perturbations About Linear Applied Field Model

We will now discuss each of these in turn

S4B: Approach 1: Explicit 3D Formulation

This is the simplest. Just employ the full 3D equations of motion expressed in terms of the applied field components $\mathbf{E}^a$, $\mathbf{B}^a$ and avoid using the focusing functions κ_x, κ_y

Comments:

- Most easy to apply in computer simulations where many effects are simultaneously included
  - Simplifies comparison to experiments when many details matter for high level agreement
- Simplifies simultaneous inclusion of transverse and longitudinal effects
  - Accelerating field E_z^a can be included to calculate changes in β_b, γ_b
  - Transverse and longitudinal dynamics cannot be fully decoupled in high level modeling – especially try when acceleration is strong in systems like injectors
- Can be applied with time based equations of motion (see: S1)
  - Helps avoid unit confusion and continuously adjusting complicated equations of motion to identify the axial coordinate s appropriately

S4C: Approach 2: Perturbations About Linear Applied Field Model

Exploit the linearity of the Maxwell equations to take:

$$\boxed{\begin{aligned}\mathbf{E}^a_\perp &= \mathbf{E}^a_\perp|_L \; + \; \delta\mathbf{E}^a_\perp \\ \mathbf{B}^a &= \mathbf{B}^a|_L \; + \; \delta\mathbf{B}^a\end{aligned}}$$

where

$\mathbf{E}^a_\perp|_L, \; \mathbf{B}^a|_L$ are the linear field components incorporated in $\kappa_x, \; \kappa_y$

to express the equations of motion as:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E^a_x - \frac{q}{m\gamma_b\beta_b c}\delta B^a_y + \frac{q}{m\gamma_b\beta_b c}\delta B^a_z y' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E^a_y + \frac{q}{m\gamma_b\beta_b c}\delta B^a_x - \frac{q}{m\gamma_b\beta_b c}\delta B^a_z x' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}\end{aligned}}$$

This formulation can be most useful to understand the effect of deviations from the usual linear model where intuition is developed

Comments:

Best suited to non-solenoidal focusing

- Simplified Larmor frame analysis for solenoidal focusing is only valid for axisymmetric potentials $\phi = \phi(r)$ which may not hold in the presence of non-ideal perturbations.
- Applied field perturbations $\delta\mathbf{E}^a_\perp, \; \delta\mathbf{B}^a$ would also need to be projected into the Larmor frame

Applied field perturbations $\delta\mathbf{E}^a_\perp, \; \delta\mathbf{B}^a$ will not necessarily satisfy the 3D Maxwell Equations by themselves

- Follows because the linear field components $\mathbf{E}^a_\perp|_L, \; \mathbf{B}^a|_L$ will not, in general, satisfy the 3D Maxwell equations by themselves

S5: Linear Transverse Particle Equations of Motion without Space-Charge, Acceleration, and Momentum Spread

S5A: Hill's Equation

Neglect:

Space-charge effects: $\partial\phi/\partial\mathbf{x} \simeq 0$

Nonlinear applied focusing and bends: $\mathbf{E}^a, \mathbf{B}^a$ have only linear focus terms

Acceleration: $\gamma_b\beta_b \simeq \text{const}$

Momentum spread effects: $v_{zi} \simeq \beta_b c$

Then the transverse particle equations of motion reduce to Hill's Equation:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

$x = \perp$ particle coordinate (i.e., x or y or possibly combinations of coordinates)

$s =$ Axial coordinate of reference particle

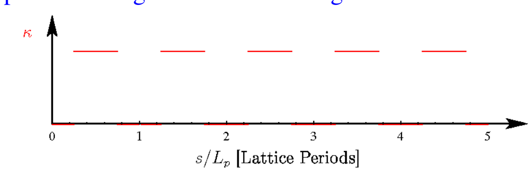
$' = \dfrac{d}{ds}$

$\kappa(s) =$ Lattice focusing function (linear fields)

For a periodic lattice:

$$\boxed{\begin{aligned}\kappa(s + L_p) &= \kappa(s) \\ L_p &= \text{Lattice Period}\end{aligned}}$$

/// Example: Hard-Edge Periodic Focusing Function



///

For a ring (i.e., circular accelerator), one also has the "superperiod" condition:

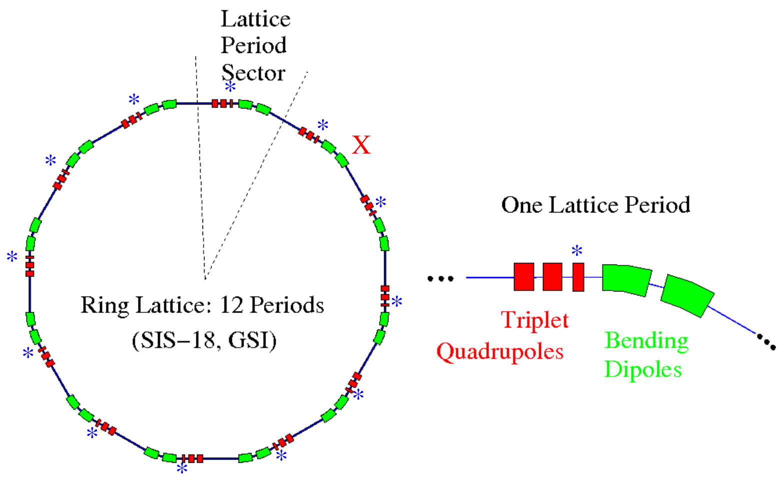
$$\boxed{\begin{aligned}\kappa(s + \mathcal{C}) &= \kappa(s) \\ \mathcal{C} = \mathcal{N}L_p &= \text{Ring Circumfrance} \\ \mathcal{N} &= \text{Superperiod Number}\end{aligned}}$$

Distinction matters when there are (field) construction errors in the ring

- Repeat with superperiod but not lattice period
- See lectures on: Particle Resonances

/// Example: Period and Superperiod distinctions for errors in a ring

* Magnet with systematic defect will be felt every lattice period
X Magnet with random (fabrication) defect felt once per lap



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S5B: Transfer Matrix Form of the Solution to Hill's Equation

Hill's equation is linear. The solution with initial condition:

$$\boxed{\begin{aligned} x(s=s_i) &= x(s_i) \\ x'(s=s_i) &= x'(s_i) \end{aligned}}$$

$s = s_i =$ Axial location of initial condition

can be uniquely expressed in matrix form (**M** is the transfer matrix) as:

$$\boxed{\begin{aligned} \begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} &= \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} \\ &= \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} \end{aligned}}$$

Where $C(s|s_i)$ and $S(s|s_i)$ are "cosine-like" and "sine-like" principal trajectories satisfying:

$$\boxed{\begin{array}{lll} C''(s|s_i) + \kappa(s) C(s|s_i) = 0 & C(s_i|s_i) = 1 & C'(s_i|s_i) = 0 \\ S''(s|s_i) + \kappa(s) S(s|s_i) = 0 & S(s_i|s_i) = 0 & S'(s_i|s_i) = 1 \end{array}}$$

Transfer matrices will be worked out in the problems for a few simple focusing systems discussed in S2 with the additional assumption of piecewise constant $\kappa(s)$

1) Drift: $\kappa = 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix}$$

2) Continuous Focusing: $\kappa = k_{\beta 0}^s = \text{const} > 0$

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[k_{\beta 0}(s-s_i)] & \frac{1}{k_{\beta 0}} \sin[k_{\beta 0}(s-s_i)] \\ -k_{\beta 0} \sin[k_{\beta 0}(s-s_i)] & \cos[k_{\beta 0}(s-s_i)] \end{bmatrix}$$

3) Solenoidal Focusing: $\kappa = \hat{\kappa} = \text{const} > 0$
Results are expressed within the rotating Larmor Frame
(same as continuous focusing with reinterpretation of variables)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sin[\sqrt{\hat{\kappa}}(s-s_i)] \\ -\sqrt{\hat{\kappa}} \sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] \end{bmatrix}$$

4) Quadrupole Focusing-Plane: $\kappa = \hat{\kappa} = \text{const} > 0$
(Obtain from continuous focusing case)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sin[\sqrt{\hat{\kappa}}(s-s_i)] \\ -\sqrt{\hat{\kappa}} \sin[\sqrt{\hat{\kappa}}(s-s_i)] & \cos[\sqrt{\hat{\kappa}}(s-s_i)] \end{bmatrix}$$

5) Quadrupole DeFocusing-Plane: $\kappa = -\hat{\kappa} = \text{const} < 0$
(Obtain from quadrupole focusing case with $\hat{\kappa} \rightarrow i\hat{\kappa}$ $\quad i = \sqrt{-1}$)

$$\mathbf{M}(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s-s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sinh[\sqrt{\hat{\kappa}}(s-s_i)] \\ \sqrt{\hat{\kappa}} \sinh[\sqrt{\hat{\kappa}}(s-s_i)] & \cosh[\sqrt{\hat{\kappa}}(s-s_i)] \end{bmatrix}$$

6) Thin Lens: $\kappa(s) = \dfrac{1}{f}\delta(s - s_0)$

$s_0 = \text{const} =$ Axial Location Lens
$f = \text{const} =$ Focal Length
$\delta(x) =$ Dirac-Delta Function

$$\mathbf{M}(s_0^+|s_0^-) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

S5C: Wronskian Symmetry of Hill's Equation

An important property of this linear motion is a Wronskian invariant/symmetry:

$$\boxed{\begin{aligned} W(s|s_i) &\equiv \det \mathbf{M}(s|s_i) = \det \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \\ &= C(s|s_i)S'(s|s_i) - C'(s|s_i)S(s|s_i) = 1 \end{aligned}}$$

/// Proof: Abbreviate Notation $C \equiv C(s|s_i)$ etc.

Multiply Equations of Motion for C and S by $-S$ and C, respectively:

$$-S(C'' + \kappa C) = 0$$
$$+C(S'' + \kappa S) = 0$$

Add Equations:

$$CS'' - SC'' + \kappa(\overbrace{CS - SC}^{0}) = 0$$
$$\Longrightarrow \frac{dW}{ds} = 0 \qquad \Longrightarrow W = \text{const}$$

Apply initial conditions:

$$W(s) = W(s_i) = C_i S_i' - C_i' S_i = 1 \cdot 1 - 0 \cdot 0 = 1$$

///

/// Example: Continuous Focusing: Transfer Matrix and Wronskian

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)]$$
$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Transfer matrix gives the familiar solution:

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \begin{bmatrix} \cos[k_{\beta 0}(s - s_i)] & \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \\ -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)] & \cos[k_{\beta 0}(s - s_i)] \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Wronskian invariant is elementary:

$$W = \cos^2[k_{\beta 0}(s - s_i)] + \sin^2[k_{\beta 0}(s - s_i)] = 1$$

///

S5D: Stability of Solutions to Hill's Equation in a Periodic Lattice

The transfer matrix must be the same in any period of the lattice:

$$\mathbf{M}(s + L_p|s_i + L_p) = \mathbf{M}(s|s_i)$$

For a propagation distance $s - s_i$ satisfying

$$NL_p \leq s - s_i \leq (N+1)L_p \qquad N = 0, 1, 2, \cdots$$

the transfer matrix can be resolved as

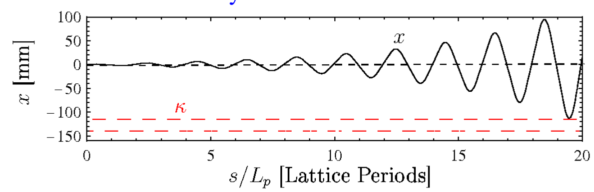
$$\boxed{\begin{aligned} \mathbf{M}(s|s_i) &= \mathbf{M}(s - NL_p|s_i) \cdot \mathbf{M}(s_i + NL_p|s_i) \\ &= \underbrace{\mathbf{M}(s - NL_p|s_i)}_{\text{Residual}} \cdot \underbrace{[\mathbf{M}(s_i + L_p|s_i)]^N}_{N \text{ Full Periods}} \end{aligned}}$$

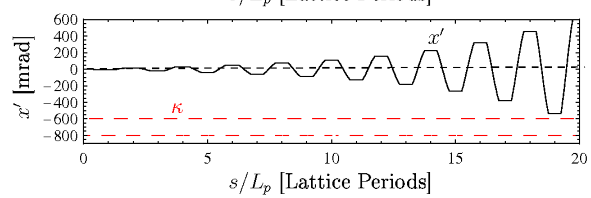
For a lattice to have stable orbits, both $x(s)$ and $x'(s)$ should remain bounded on propagation through an arbitrary number N of lattice periods. This is equivalent to requiring that the elements of $\mathbf{M}$ remain bounded on propagation through any number of lattice periods:

$$\mathbf{M}^N \equiv [\mathbf{M}^N{}_{ij}]$$

$$\boxed{\lim_{N \to \infty} \left|\mathbf{M}^N{}_{ij}\right| < \infty \quad \Longrightarrow \text{Stable Motion}}$$

Clarification of stability notion: Unstable Orbit





$$L_p = 0.5 \text{ m}$$
$$\eta = 0.5$$
$$\kappa = \begin{cases} 48 & \text{where } \kappa \neq 0 \\ 0 & \text{otherwise} \end{cases}$$
$$x(0) = 1 \text{ mm}$$
$$x'(0) = 0$$

For energetic particle:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \quad \sim \text{Large, but } \neq \text{const}$$

where $|x'|$ small, $|x|$ large

where $|x|$ small, $|x'|$ large

The matrix criterion corresponds to our intuitive notion of stability: as the particle advances there are no large oscillation excursions in position and angle.

To analyze the stability condition, examine the eigenvectors/eigenvalues of $\mathbf{M}$ for transport through one lattice period:

$$\boxed{\begin{aligned} &\mathbf{M}(s_i + L_p|s_i) \cdot \mathbf{E} \equiv \lambda \mathbf{E} \\ &\mathbf{E} = \text{Eigenvector} \\ &\lambda = \text{Eigenvalue} \end{aligned}}$$

Eigenvectors and Eigenvalues are generally complex
Eigenvectors and Eigenvalues generally vary with s_i
Two independent Eigenvalues and Eigenvectors
- Degeneracies special case

Derive the two independent eigenvectors/eigenvalues through analysis of the characteristic equation: Abbreviate Notation

$$\mathbf{M}(s_i + L_p|s_i) = \begin{bmatrix} C(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & S'(s_i + L_p|s_i) \end{bmatrix} \equiv \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

Nontrivial solutions exist when:

$$\det \begin{bmatrix} C - \lambda & S \\ C' & S' - \lambda \end{bmatrix} = \lambda^2 - (C + S')\lambda + (CS' - SC') = 0$$

But we can apply the Wronskian condition:

$$CS' - SC' = 1$$

and we make the notational definition

$$C + S' = \text{Tr}\,\mathbf{M} \equiv 2\cos\sigma_0$$

The characteristic equation then reduces to:

$$\boxed{\lambda^2 - 2\lambda\cos\sigma_0 + 1 = 0 \qquad \cos\sigma_0 \equiv \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

The use of $2\cos\sigma_0$ to denote Tr $\mathbf{M}$ is in anticipation of later results (see S6) where σ_0 is identified as the phase-advance of a stable orbit

There are two solutions to the characteristic equation that we denote $\lambda_\pm$

$$\boxed{\begin{aligned} &\lambda_\pm = \cos\sigma_0 \pm \sqrt{\cos^2\sigma_0 - 1} = \cos\sigma_0 \pm i\sin\sigma_0 = e^{\pm i\sigma_0} \\ &\mathbf{E}_\pm = \text{Corresponding Eigenvectors} \qquad i \equiv \sqrt{-1} \end{aligned}}$$

Note that: $\lambda_+\lambda_- = 1$
$\lambda_+ = 1/\lambda_-$

Consider a vector of initial conditions:

$$\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} x_i \\ x'_i \end{bmatrix}$$

The eigenvectors $\mathbf{E}_\pm$ span two-dimensional space. So any initial condition vector can be expanded as:

$$\boxed{\begin{aligned} &\begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+ \mathbf{E}_+ + \alpha_- \mathbf{E}_- \\ &\alpha_\pm = \text{Complex Constants} \end{aligned}}$$

Then using $\mathbf{M}\mathbf{E}_\pm = \lambda_\pm \mathbf{E}_\pm$

$$\boxed{\mathbf{M}^N(s_i + L_p|s_i)\begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+\lambda_+^N\mathbf{E}_+ + \alpha_-\lambda_-^N\mathbf{E}_-}$$

Therefore, if $\lim_{N\to\infty} \lambda^N$ is bounded, then the motion is stable. This will always be the case if $|\lambda_\pm| \leq 1$, corresponding to σ_0 real with $|\cos\sigma_0| \leq 1$

This implies for stability or the orbit that we must have:

$$\boxed{\begin{aligned} \frac{1}{2}|\text{Trace}\,\mathbf{M}(s_i + L_p|s_i)| &= \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)| \\ &= |\cos\sigma_0| \leq 1 \end{aligned}}$$

In a periodic focusing lattice, this important stability condition places restrictions on the lattice structure (focusing strength) that are generally interpreted in terms of phase advance limits (see: S6).

Accelerator lattices almost always tuned for single particle stability to maintain beam control
- Even for intense beams, beam centroid approximately obeys single particle equations of motion when image charges are negligible

Space-charge and nonlinear applied fields can further limit particle stability
- Resonances: see: Particle Resonances ....
- Envelope Instability: see: Transverse Centroid and Envelope ....
- Higher Order Instability: see: Transverse Kinetic Stability

We will show (see: S6) that for stable orbits σ_0 can be interpreted as the phase-advance of single particle oscillations

/// Example: Continuous Focusing Stability

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Principal orbit equations are simple harmonic oscillators with solution:

$$C(s|s_i) = \cos[k_{\beta 0}(s - s_i)] \qquad C'(s|s_i) = -k_{\beta 0}\sin[k_{\beta 0}(s - s_i)]$$

$$S(s|s_i) = \frac{\sin[k_{\beta 0}(s - s_i)]}{k_{\beta 0}} \qquad S'(s|s_i) = \cos[k_{\beta 0}(s - s_i)]$$

Stability bound then gives:

$$\frac{1}{2}|\text{Trace}\,\mathbf{M}(s_i + L_p|s_i)| = \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)|$$

$$= |\cos(k_{\beta 0}(s - s_i)| \leq 1$$

Always satisfied for real $k_{\beta 0}$

Confirms known result using formalism: continuous focusing stable

- Energy not pumped into or out of particle orbit ///

The simplest example of the stability criterion applied to periodic lattices will be given in the problem sets: Stability of a periodic thin lens lattice

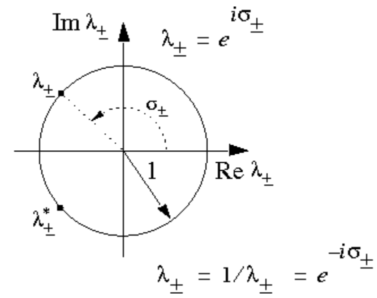
Analytically find that lattice unstable when focusing kicks sufficiently strong

More advanced treatments

See: Dragt, *Lectures on Nonlinear Orbit Dynamics*, AIP Conf Proc 87 (1982)

show that symplectic 2x2 transfer matrices associated with Hill's Equation have only two possible classes of eigenvalue symmetries:

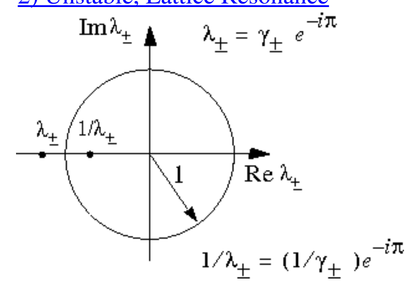
1) Stable



Occurs for:

$$0 \leq \sigma_0 \leq 180^\circ/\text{period}$$

2) Unstable, Lattice Resonance



Occurs in bands when focusing strength is increased beyond

$$\sigma_0 = 180^\circ/\text{period}$$

Limited class of possibilities simplifies analysis of focusing lattices

S6: Hill's Equation: Floquet's Theorem and the Phase-Amplitude Form of the Particle Orbit

S6A: Introduction

In this section we consider Hill's Equation:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

subject to a periodic applied focusing function

$$\boxed{\begin{array}{l}\kappa(s + L_p) = \kappa(s)\\ L_p = \text{Lattice Period}\end{array}}$$

Many results will also hold in more complicated form for a non-periodic $\kappa(s)$

S6B: Floquet's Theorem

Floquet's Theorem (proof: see standard Mathematics and Mathematical Physics Texts)

The solution to Hill's Equation *x*(*s*) has two linearly independent solutions that can be expressed as:

$$i = \sqrt{-1}$$

$$x_1(s) = w(s)e^{i\mu s}$$

$$x_2(s) = w(s)e^{-i\mu s}$$

$$\mu = \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i) = \cos\sigma_0$$

$$= \text{const} = \text{Characteristic Exponent}$$

Where *w*(*s*) is a periodic function:

$$w(s + L_p) = w(s)$$

Theorem as written only applies for **M** with non-degenerate eigenvalues. But a similar theorem applies in the degenerate case.

A similar theorem is also valid for non-periodic focusing functions

S6C: Phase-Amplitude Form of Particle Orbit

As a consequence of Floquet's Theorem, any (stable or unstable) nondegenerate solution to Hill's Equation can be expressed in phase-amplitude form as:

$$\boxed{\begin{aligned} x(s) &= A(s)\cos\psi(s) & A(s) &= \text{Amplitude Function} \\ A(s+L_p) &= A(s) & \psi(s) &= \text{Phase Function} \end{aligned}}$$

Derive equations of motion for A, ψ by taking derivatives of the phase-amplitude form for $x(s)$:

$$x = A\cos\psi$$
$$x' = A'\cos\psi - A\psi'\sin\psi$$
$$x'' = A''\cos\psi - 2A'\psi'\sin\psi - A\psi''\sin\psi - A\psi'^2\cos\psi$$

then substitute in Hill's Equation:

$$\boxed{x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0}$$

$$\boxed{x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0}$$

We are free to introduce an additional constraint between A and ψ:

Two functions A, ψ to represent one function x allows a constraint

Choose:

Eq. (1) $\boxed{2A'\psi' + A\psi'' = 0}$ $\Longrightarrow$ Coefficient of $\sin\psi$ zero

Then to satisfy Hill's Equation for all ψ, the coefficient of $\cos\psi$ must also vanish giving:

Eq. (2) $\boxed{A'' + \kappa A - A\psi'^2 = 0}$ $\Longrightarrow$ Coefficient of $\cos\psi$ zero

Eq. (1) Analysis (coefficient of $\sin\psi$): $2A'\psi' + A\psi'' = 0$

Simplify:

$$2A'\psi' + A\psi'' = \frac{(A^2\psi')'}{A} = 0 \qquad A \neq 0$$

Will show later that this assumption met for all s

$$\Longrightarrow \quad (A^2\psi')' = 0$$

Integrate once:

$$\boxed{A^2\psi' = \text{const}}$$

One commonly rescales the amplitude $A(s)$ in terms of an auxiliary amplitude function $w(s)$:

$$\boxed{A(s) = A_i w(s) \qquad A_i = \text{const} = \text{Initial Amplitude}}$$

such that

$$\boxed{w^2\psi' \equiv 1}$$

This equation can then be integrated to obtain the phase-function of the particle:

$$\boxed{\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \psi_i = \text{const} = \text{Initial Phase}}$$

Eq. (2) Analysis (coefficient of $\cos\psi$): $A'' + \kappa A - A\psi'^2 = 0$

With the choice of amplitude rescaling, $w^2\psi' = 1$ and Eq. (2) becomes:

$$\boxed{w'' + \kappa w - \frac{1}{w^3} = 0}$$

Floquet's theorem tells us that we are free to restrict w to be a periodic solution:

$$\boxed{w(s + L_p) = w(s)}$$

Reduced Expressions for x and x':

Using $A = A_i w$ and $w^2\psi' = 1$:

$$x = A\cos\psi$$
$$x' = A'\cos\psi - A\psi'\sin\psi$$

$$\Longrightarrow \quad \boxed{\begin{aligned} x &= A_i w\cos\psi \\ x' &= A_i w'\cos\psi - \frac{A_i}{w}\sin\psi \end{aligned}}$$

S6D: Summary: Phase-Amplitude Form of Solution to Hill's Eqn

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s)$$

$A_i = \text{const} = \text{Initial Amplitude}$

$\psi_i = \text{const} = \text{Initial Phase}$

where $w(s)$ and $\psi(s)$ are amplitude- and phase-functions satisfying:

Amplitude Equations

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s+L_p) = w(s)$$

$$w(s) > 0$$

Phase Equations

$$\psi'(s) = \frac{1}{w^2(s)}$$

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$\psi(s) = \psi_i + \Delta\psi(s)$$

Initial ($s = s_i$) amplitudes are constrained by the particle initial conditions as:

$$x(s=s_i) = A_i w_i \cos\psi_i$$

$$x'(s=s_i) = A_i w_i' \cos\psi_i - \frac{A_i}{w_i}\sin\psi_i$$

or

$$A_i \cos\psi_i = x(s=s_i)/w_i \qquad w_i \equiv w(s=s_i)$$

$$A_i \sin\psi_i = x(s=s_i)w_i' - x'(s=s_i)w_i \qquad w_i' \equiv w'(s=s_i)$$

S6E: Points on the Phase-Amplitude Formulation

1) $w(s)$ can be taken as positive definite

$$w(s) > 0$$

/// Proof: Sign choices in w:

Let $w(s)$ be positive at some point. Then the equation:

$$w'' + \kappa w - \frac{1}{w^3} = 0$$

Insures that w can never vanish or change sign. This follows because whenever w becomes small, $w'' \simeq 1/w^3 \gg 0$ can become arbitrarily large to turn w before it reaches zero. Thus, to fix phases, we conveniently require that $w > 0$. ///

- Proof verifies assumption made in analysis that $A = A_i w \neq 0$
- Conversely, one could choose w negative and it would always remain negative for analogous reasons. This choice is *not* commonly made.
- Sign choice removes ambiguity in relating initial conditions $x(s_i),\ x'(s_i)$ to $A_i,\ \psi_i$

2) $w(s)$ is a unique periodic function
- Can be proved using a connection between w and the principal orbit functions C and S (see: Appendix C and S7)
- $w(s)$ can be regarded as a special, periodic function describing the lattice

3) The amplitude parameters

$$w_i = w(s=s_i)$$

$$w_i' = w'(s_i)$$

depend *only* on the periodic lattice properties and are *independent* of the particle initial conditions $x(s_i),\ x'(s_i)$

4) The change in phase

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

depends on the choice of initial condition s_i. However, the phase-advance through one lattice period

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

is independent of s_i since w is a periodic function with period L_p
- Will show that (see later in this section)

$$\Delta\psi(s_i+L_p) \equiv \sigma_0$$

is the undepressed phase advance of particle oscillations

5) $w(s)$ has dimensions [[w]] = Sqrt[meters]
- Can prove inconvenient in applications and motivates the use of an alternative "betatron" function β

$$\beta(s) \equiv w^2(s)$$

with dimension [[β]] = meters (see: S7 and S8)

6) On the surface, what we have done: Transform the linear Hill's Equation to a form where a solution to nonlinear axillary equations for w and ψ are needed via the phase-amplitude method seems insane ..... why do it?
- Method will help identify the useful Courant-Snyder invariant which will aid interpretation of the dynamics (see: S7)
- Decoupling of initial conditions in the phase-amplitude method will help simplify understanding of bundles of particles in the distribution

S6F: Relation between Principal Orbit Functions and Phase-Amplitude Form Orbit Functions

The transfer matrix **M** of the particle orbit can be expressed in terms of the principal orbit functions *C* and *S* as (see: S4):

$$\begin{bmatrix} x(s) \\ x'(s) \end{bmatrix} = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix}$$

Use of the phase-amplitude forms and some algebra identifies (see problem sets):

$$C(s|s_i) = \frac{w(s)}{w_i}\cos\Delta\psi(s) - w_i' w(s)\sin\Delta\psi(s)$$

$$S(s|s_i) = w_i w(s)\sin\Delta\psi(s)$$

$$C'(s|s_i) = \left(\frac{w'(s)}{w_i} - \frac{w_i'}{w(s)}\right)\cos\Delta\psi(s) - \left(\frac{1}{w_i w(s)} + w_i' w'(s)\right)\sin\Delta\psi(s)$$

$$S'(s|s_i) = \frac{w_i}{w(s)}\cos\Delta\psi(s) + w_i w'(s)\sin\Delta\psi(s)$$

$$\Delta\psi(s) \equiv \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \begin{aligned} w_i &\equiv w(s=s_i) \\ w_i' &\equiv w'(s=s_i) \end{aligned}$$

/// Aside: Alternatively, it can be shown (see: Appendix C) that *w*(*s*) can be related to the principal orbit functions calculated over one Lattice period by:

$$w^2(s) = \beta(s) = \sin\sigma_0 \frac{S(s|s_i)}{S(s_i+L_p|s_i)} + \frac{S(s_i+L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s|s_i)}{S(s_i+L_p|s_i)}S(s|s_i)\right]^2$$

$$\sigma_0 \equiv \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

The formula for σ_0 in terms of principal orbit functions is useful:

σ_0 (phase advance, see: S6G) is often specified for the lattice and the focusing function $\kappa(s)$ is tuned to achieve the specified value

Shows that *w*(*s*) can be constructed from two principal orbit integrations over one lattice period

- Integrations must generally be done numerically for *C* and *S*
- No root finding required for initial conditions to construct periodic *w*(*s*)
- s_i can be anywhere in the lattice period and *w*(*s*) will be independent of the specific choice of s_i

The form of $w^2(s)$ suggests an underlying Courant-Snyder Invariant (see: S7 and Appendix C)

$w^2 = \beta$ can be applied to calculate max beam particle excursions in the absence of space-charge effects (see: S8)

- Useful in machine design
- Exploits Courant-Snyder Invariant

///

S6G: Undepressed Particle Phase Advance

We can now concretely connect σ_0 for a stable orbit to the change in particle oscillation phase $\Delta\psi$ through one lattice period:

From S5D:

$$\cos\sigma_0 \equiv \frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i+L_p|s_i)$$

Apply the principal orbit representation of **M**

$$\mathrm{Tr}\,\mathbf{M}(s_i+L_p|s_i) = C(s_i+L_p|s_i) + S'(s_i+L_p|s_i)$$

and use the phase-amplitude identifications of *C* and *S*' calculated in S6F:

$$\frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i+L_p|s_i) = \frac{1}{2}\left(\frac{w(s_i+L_p)}{w_i} + \frac{w_i}{w(s_i+L_p)}\right)\cos\Delta\psi(s_i+L_p) + \frac{1}{2}\left(w_i w'(s_i+L_p) - w_i' w(s_i+L_p)\right)\sin\Delta\psi(s_i+L_p)$$

By periodicity:

$$\begin{aligned} w(s_i+L_p) &= w(s_i) = w_i \\ w'(s_i+L_p) &= w'(s_i) = w_i' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} &\text{coefficient of } \cos\Delta\psi = 1 \\ &\text{coefficient of } \sin\Delta\psi = 0 \end{aligned}$$

Applying these results gives:

$$\boxed{\cos\sigma_0 = \cos\Delta\psi(s_i + L_p) = \frac{1}{2}\mathrm{Tr}\ \mathbf{M}(s_i + L_p|s_i)}$$

Thus, σ_0 is identified as the phase advance of a stable particle orbit through one lattice period:

$$\boxed{\sigma_0 = \Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)}}$$

Again verifies that σ_0 is independent of s_i since w(s) is periodic with period L_p

The stability criterion (see: S5)

$$\frac{1}{2}|\mathrm{Tr}\ \mathbf{M}(s_i + L_p|s_i)| = |\cos\sigma_0| \leq 1$$

is concretely connected to the particle phase advance through one lattice period providing a useful physical interpretation

Consequence:

Any periodic lattice with undepressed phase advance satisfying

$$\sigma_0 < \pi/\text{period} = 180^\circ/\text{period}$$

will have stable single particle orbits.

Discussion:

The phase advance σ_0 is an extremely useful dimensionless measure to characterize the focusing strength of a periodic lattice. Much of conventional accelerator physics centers on focusing strength and the suppression of resonance effects. The phase advance is a natural parameter to employ in many situations to allow ready interpretation of results in a generalizable manner.

We present phase advance formulas for several simple classes of lattices to help build intuition on focusing strength:

1) Continuous Focusing
2) Periodic Solenoidal Focusing
3) Periodic Quadrupole Doublet Focusing
   - FODO Quadrupole Limit

Several of these will be derived in the problem sets

Lattices analyzed as "hard-edge" with piecewise-constant $\kappa(s)$ and lattice period L_p

Results are summarized only with derivations guided in the problem sets.

4) Thin Lens Limits
   - Useful for analysis of scaling properties

1) Continuous Focusing

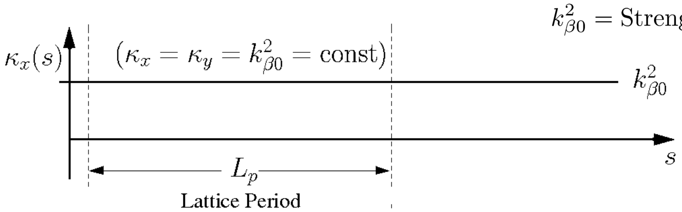
"Lattice period" L_p is an arbitrary length for phase accumulation

$$\kappa(s) = k_{\beta 0}^2 = \text{const} > 0$$

Parameters:

L_p = Lattice Period

$k_{\beta 0}^2$ = Strength



Apply phase advance formulas:

$$w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0 \quad \Longrightarrow \quad w = \frac{1}{\sqrt{k_{\beta 0}}}$$

$$\sigma_0 = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2} = k_{\beta 0} L_p$$

$$\boxed{\sigma_0 = k_{\beta 0} L_p}$$

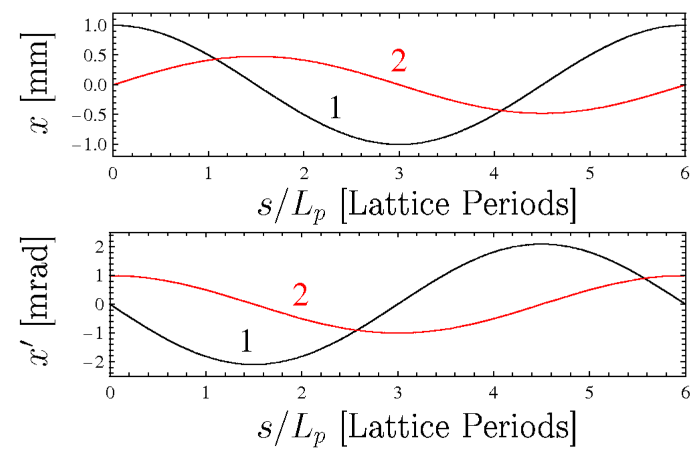
Always stable
- Energy cannot pump into or out of particle orbit

Rescaled Principal Orbit Evolution:

$L_p = 0.5$ m
$\sigma_0 = \pi/3 = 60^\circ$
$k_{\beta 0} = (\pi/6)$ rad/m

Cosine-Like
1: $x(0) = 1$ mm
$x'(0) = 0$ mrad

Sine-Like
2: $x(0) = 0$ mm
$x'(0) = 1$ mrad



Phase-Space Evolution (see also S7):

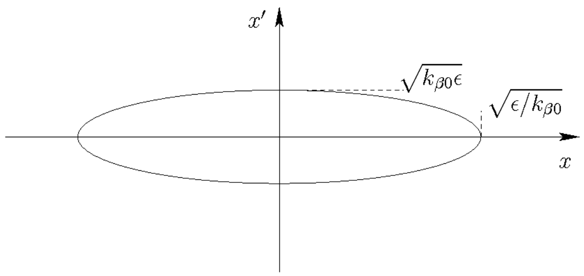
Phase-space ellipse stationary and aligned along *x*, *x'* axes for continuous focusing

$$w = \sqrt{1/k_{\beta 0}} = \text{const} \qquad \gamma = \frac{1}{w^2} = k_{\beta 0} = \text{const}$$

$$w' = 0 \qquad \alpha = -ww' = 0$$

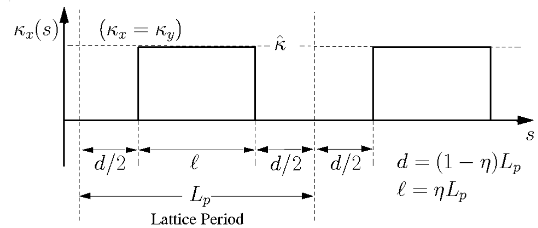
$$\beta = w^2 = 1/k_{\beta 0} = \text{const}$$

$$k_{\beta 0} x^2 + x'^2/k_{\beta 0} = \epsilon = \text{const}$$



2) Periodic Solenoidal Focusing

Results are interpreted in the rotating Larmor frame (see S2 and Appendix A)



Parameters:

L_p = Lattice Period

$\eta \in (0, 1]$ = Occupancy

$\hat{\kappa}$ = Strength

Characteristics:

ηL_p = Optic Length

$(1-\eta)L_p$ = Drift Length

Calculation gives:

$$\boxed{\cos\sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta}\Theta\sin(2\Theta) \qquad \Theta \equiv \frac{\eta}{2}\sqrt{\hat{\kappa}}L_p}$$

Can be unstable when $\hat{\kappa}$ becomes large

- Energy can pump into or out of particle orbit

Rescaled Larmor-Frame Principal Orbit Evolution Solenoid Focusing:

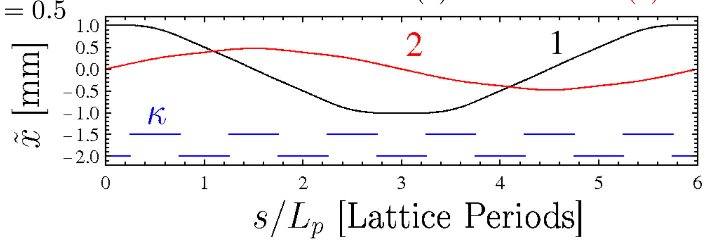
$L_p = 0.5$ m

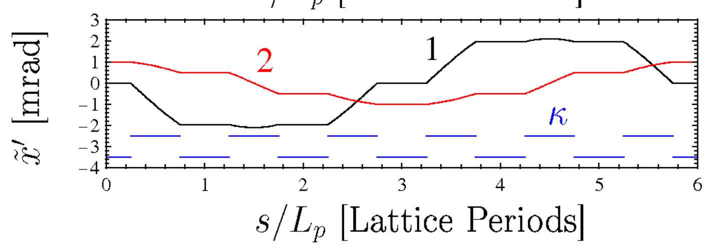
$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 8.558$ m^{-2})

$\eta = 0.5$

Cosine-Like
1: $\tilde{x}(0) = 1$ mm
$\tilde{x}'(0) = 0$ mrad

Sine-Like
2: $\tilde{x}(0) = 0$ mm
$\tilde{x}'(0) = 1$ mrad





Principal orbits in $\tilde{y} - \tilde{y}'$ phase-space are identical

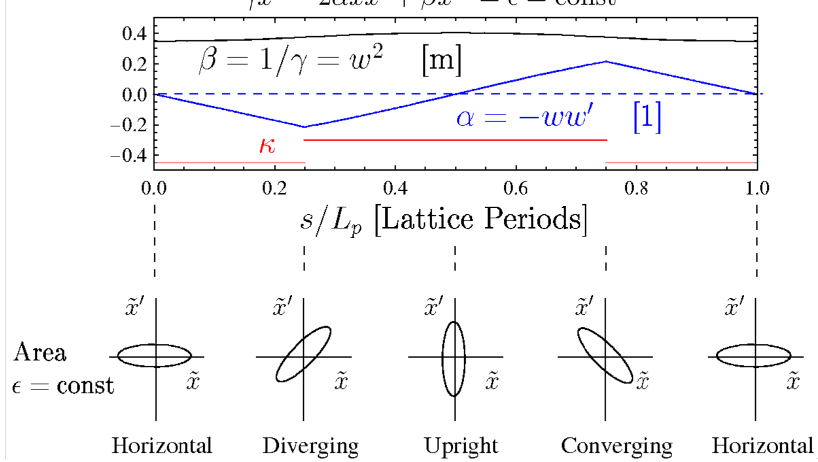
Phase-Space Evolution in the Larmor frame (see also: S7):

Phase-Space ellipse rotates and evolves in periodic lattice

$\tilde{y} - \tilde{y}'$ phase-space properties same as in $\tilde{x} - \tilde{x}'$

- Phase-space structure in *x*-*x'*, *y*-*y'* phase space is complicated

$$\gamma\tilde{x}^2 - 2\alpha\tilde{x}\tilde{x}' + \beta\tilde{x}'^2 = \epsilon = \text{const}$$



Comments on periodic solenoid results:

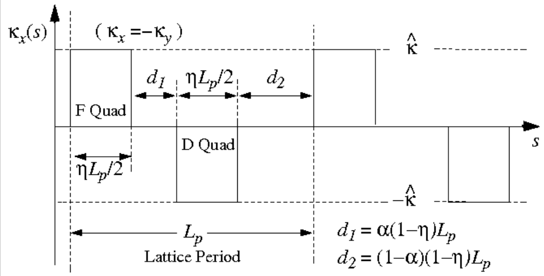
Larmor frame analysis greatly simplifies results

- 4D coupled orbit in x-x', y-y' phase-space will be much more intricate in structure

Phase-Space ellipse rotates and evolves in periodic lattice

Periodic structure of lattice changes orbits from simple harmonic

3) Periodic Quadrupole Doublet Focusing



Parameters:

$L_p =$ Lattice Period

$\eta \in (0,1] =$ Occupancy

$\alpha \in [0,1] =$ Syncopation

$\hat{\kappa} =$ Strength

Characteristics:

$\eta L_p/2 =$ F/D Len

$\alpha(1-\eta)L_p =$ Drift Len d_1

$(1-\alpha)(1-\eta)L_p =$ Drift Len d_2

Calculation gives:

$$\boxed{\begin{aligned}\cos\sigma_0 &= \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) \\ &\quad - 2\alpha(1-\alpha)\frac{(1-\eta)^2}{\eta^2}\Theta^2\sin\Theta\sinh\Theta \qquad \Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p\end{aligned}}$$

Can be unstable when $\hat{\kappa}$ becomes large

- Energy can pump into or out of particle orbit

Comments on Parameters:

The "syncopation" parameter α measures how close the Focusing (F) and DeFocusing (D) quadrupoles are to each other in the lattice

$$\alpha \in [0,1] \qquad \begin{aligned} \alpha = 0 &\implies d_1 = 0 \qquad d_2 = (1-\eta)L_p \\ \alpha = 1 &\implies d_1 = (1-\eta)L_p \quad d_2 = 0 \end{aligned}$$

The range $\alpha \in [1/2, 1]$ can be mapped to $\alpha \in [0, 1/2]$ by simply relabeling quantities. Therefore, we can take:

$$\alpha \in [0, 1/2]$$

The special case of a doublet lattice with $\alpha = 1/2$ corresponds to equal drift lengths between the F and D quadrupoles and is called a FODO lattice

$$\alpha = 1/2 \implies d_1 = d_2 \equiv d = (1-\eta)L_p/2$$

Phase advance constraint will be derived for FODO case in problems (algebra much simpler than doublet case)

Special Case Doublet Focusing: Periodic Quadrupole FODO Lattice

Parameters:

$L_p =$ Lattice Period

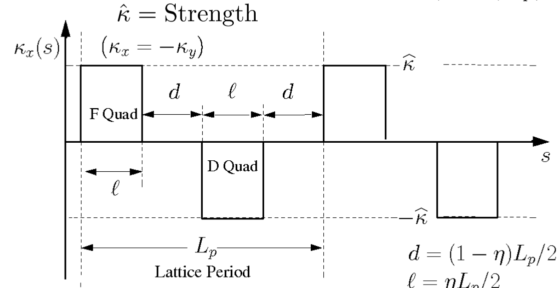
$\eta \in (0,1] =$ Occupancy

$\hat{\kappa} =$ Strength

Characteristics:

$\eta L_p/2 = \ell =$ F/D Len

$(1-\eta)L_p/2 = d =$ Drift Len



Phase advance formula reduces to:

$$\boxed{\begin{aligned}\cos\sigma_0 &= \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) \\ &\quad - \frac{(1-\eta)^2}{2\eta^2}\Theta^2\sin\Theta\sinh\Theta \qquad \Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p\end{aligned}}$$

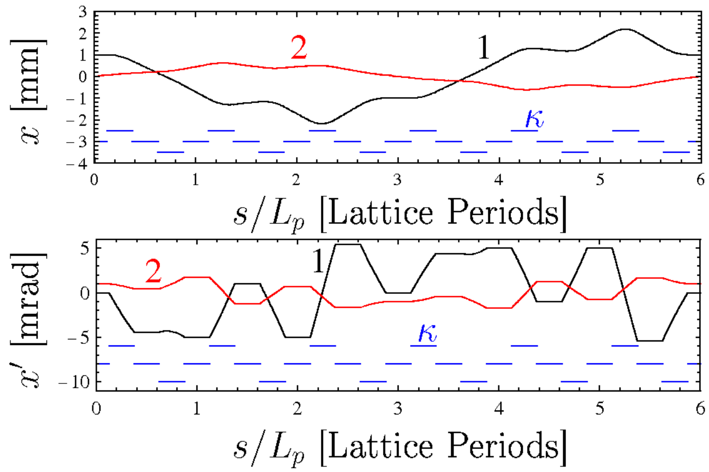
Analysis shows FODO provides stronger focus for same integrated field gradients than doublet due to symmetry

Rescaled Principal Orbit Evolution FODO Quadrupole:

$L_p = 0.5$ m
$\sigma_0 = \pi/3 = 60^\circ$ $(\kappa = 39.24\ \mathrm{m}^{-2})$
$\eta = 0.5$

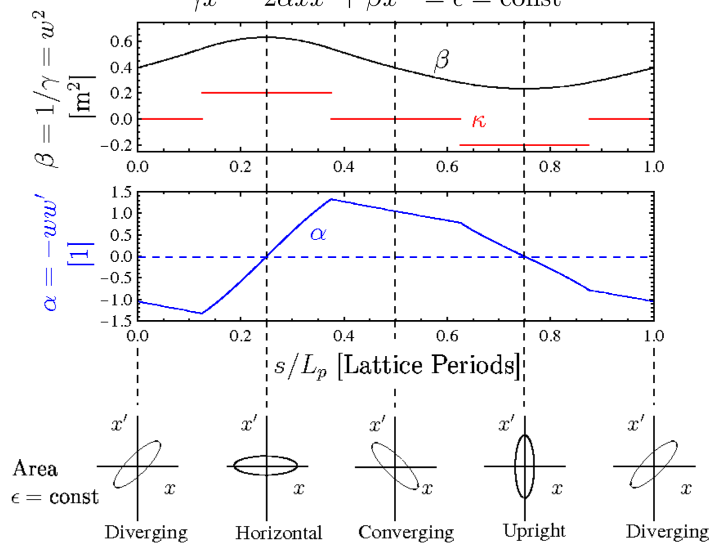
Cosine-Like
1: $x(0) = 1$ mm
$x'(0) = 0$ mrad

Sine-Like
2: $x(0) = 0$ mm
$x'(0) = 1$ mrad



Phase-Space Evolution (see also: S7):

$$\gamma x^2 - 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}$$



Comments on periodic FODO quadrupole results:

Phase-Space ellipse rotates and evolves in periodic lattice
- Evolution more intricate for Alternating Gradient (AG) focusing than for solenoidal focusing in the Larmor frame

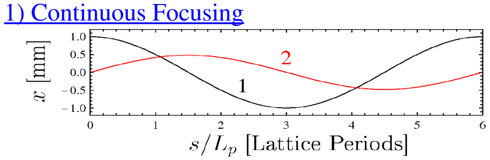
Harmonic content of orbits larger for AG focusing than solenodial focusing

Orbit and phase space evolution analogous in *y-y'* plane
- Simply related by an shift in *s* of the lattice

Contrast of Principal Orbits for different focusing:

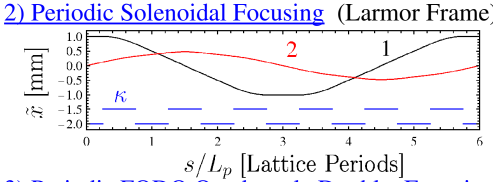
Use previous examples with "equivalent" focusing strength $\sigma_0 = 60^\circ$
Note that periodic focusing adds harmonic structure

1) Continuous Focusing



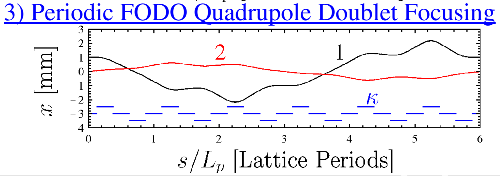
Simple Harmonic Oscillator

2) Periodic Solenoidal Focusing (Larmor Frame)



Simple harmonic oscillations modified with additional harmonics due to periodic focus

3) Periodic FODO Quadrupole Doublet Focusing



Simple harmonic oscillations more strongly modified due to periodic AG focus

4) Thin Lens Limits

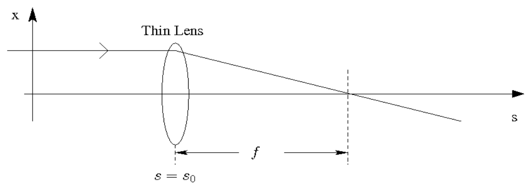
Convenient to simply understand analytic scaling

$$\boxed{\begin{aligned} \kappa_x(s) &= \frac{1}{f}\delta(s-s_0) \\ s_0 &= \text{Optic Location} = \text{const} \\ f &= \text{focal length} = \text{const} \end{aligned}}$$

Transfer Matrix:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^+} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s=s_0^-}$$

Graphical Interpretation:



The thin lens limit of "thick" hard-edge solenoid and quadrupole focusing lattices presented can be obtained by taking:

Solenoids: $\hat{\kappa} \equiv \dfrac{1}{\eta f L_p}$ then take $\lim_{\eta\to 0}$

Quadrupoles: $\hat{\kappa} \equiv \dfrac{2}{\eta f L_p}$ then take $\lim_{\eta\to 0}$

This obtains when applied in the previous formulas:

$$\boxed{\cos\sigma_0 = \begin{cases} 1 - \frac{1}{2}\frac{L_p}{f}, & \text{thin-lens periodic solenoid} \\ 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2, & \text{thin-lens quadrupole doublet} \\ & \alpha = \frac{1}{2} \Longrightarrow \text{FODO} \end{cases}}$$

These formulas can also be derived directly from the drift and thin lens transfer matrices as

Periodic Solenoid

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = 1 - \frac{1}{2}\frac{L_p}{f}$$

Periodic Quadrupole Doublet

$$\cos\sigma_0 = \frac{1}{2}\mathrm{Tr}\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & \alpha L_p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{bmatrix}\begin{bmatrix} 1 & (1-\alpha)L_p \\ 0 & 1 \end{bmatrix} = 1 - \frac{\alpha}{2}(1-\alpha)\left(\frac{L_p}{f}\right)^2$$

Expanded phase advance formulas (thin lens type limit and similar) can be useful in system design studies

- Desirable to derive simple formulas relating magnet parameters to σ_0
  - Clear analytic scaling trends clarify design trade-offs
- For hard edge periodic lattices, expand formula for $\cos\sigma_0$ to leading order in $\Theta = \sqrt{|\hat{\kappa}|}\eta L_p/2$

/// Example: Periodic Quadrupole Doublet Focusing:

Expand previous formula

$$\cos\sigma_0 = 1 - \frac{(\eta\hat{\kappa}L_p^2)^2}{32}\left[\left(1-\frac{2}{3}\eta\right) - 4\left(\alpha - \frac{1}{2}\right)^2(1-\eta)^2\right]$$

where:

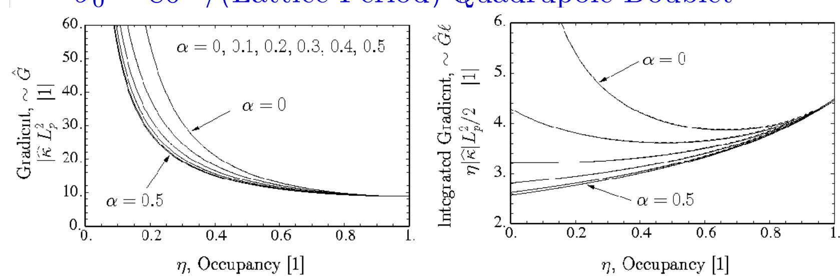
$$\hat{\kappa} = \begin{cases} \frac{\hat{G}}{[B\rho]}, & \text{Magnetic Quadrupoles} \\ \frac{\hat{G}}{\beta_b c[B\rho]}, & \text{Electric Quadrupoles} \end{cases} \qquad \hat{G} = \text{Hard-Edge Field Gradient}$$

Using these results, plot the Field Gradient and Integrated Gradient for quadrupole doublet focusing needed for $\sigma_0 = 80^\circ$ per lattice period

Gradient ~ $|\hat{\kappa}|L_p^2 \sim \hat{G}$

Integrated Gradient ~ $\eta|\hat{\kappa}|L_p^2/2 \sim \hat{G}\ell$

$\sigma_0 = 80^\circ$ /(Lattice Period) Quadrupole Doublet



- Exact (non-expanded) solutions plotted dashed (almost overlay)
- Gradient and integrated gradient required depend only weakly on syncopation factor α when α is near ½
- Stronger gradient required for low occupancy η but integrated gradient varies little with η

///

Appendix C: Calculation of *w*(*s*) from Principal Orbit Functions

Evaluate principal orbit expressions of the transfer matrix through one lattice period using

$$w(s_i + L_p) = w_i$$
$$w'(s_i + L_p) = w'_i$$

and

$$\Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)} = \sigma_0$$

to obtain (see principal orbit formulas expressed in phase-amplitude form):

$$C(s_i + L_p|s_i) = \cos\sigma_0 - w_i w'_i \sin\sigma_0$$
$$S(s_i + L_p|s_i) = w_i^2 \sin\sigma_0$$
$$C'(s_i + L_p|s_i) = -\left(\frac{1}{w_i^2} + w_i w'_i\right)\sin\sigma_0$$
$$S'(s_i + L_p|s_i) = \cos\sigma_0 + w_i w'_i \sin\sigma_0$$

C1

Giving:

$$\boxed{\begin{aligned} w_i &= \sqrt{\frac{S(s_i + L_p|s_i)}{\sin\sigma_0}} \\ w'_i &= \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{\sqrt{S(s_i + L_p|s_i)\sin\sigma_0}} \end{aligned}}$$

Or in terms of the betatron formulation (see: S7 and S8) with

$$\beta = w^2, \ \beta' = 2ww'$$

$$\boxed{\begin{aligned} \beta_i &= w_i^2 = \frac{S(s_i + L_p|s_i)}{\sin\sigma_0} \\ \beta'_i &= 2w_i w'_i = \frac{2[\cos\sigma_0 - C(s_i + L_p|s_i)]}{\sin\sigma_0} \end{aligned}}$$

Next, calculate w from the principal orbit expression in phase-amplitude form:

$$\frac{S}{w_i w} = \sin\Delta\psi$$
$$\frac{w_i}{w}C + \frac{w'_i}{w}S = \cos\Delta\psi$$

$$S \equiv S(s|s_i) \ \ \text{etc.}$$

C2

Square and add equations:

$$\left(\frac{S}{w_i w}\right)^2 + \left(\frac{w_i C}{w} + \frac{w'_i S}{w}\right)^2 = 1$$

This result reflects the structure of the underlying Courant-Snyder invariant (see: S7)

Gives:

$$w^2 = \left(\frac{S}{w_i}\right)^2 + (w_i C + w'_i S)^2$$

Use $w_i, \ w'_i$ previously identified and write out result:

$$\boxed{\begin{aligned} w^2(s) = \beta(s) = \sin\sigma_0 & \frac{S^2(s|s_i)}{S(s_i + L_p|s_i)} \\ & + \frac{S(s_i + L_p|s_i)}{\sin\sigma_0}\left[C(s|s_i) + \frac{\cos\sigma_0 - C(s_i + L_p|s_i)}{S(s_i + L_p|s_i)}S(s|s_i)\right]^2 \end{aligned}}$$

Formula shows that for a given σ_0 (used to specify lattice focusing strength), *w*(*s*) is given by two linear principal orbits calculated over one lattice period
- Easy to apply numerically

C3

An alternative way to calculate *w*(*s*) is as follows. 1st apply the phase-amplitude formulas for the principal orbit functions with:

$$s_i \rightarrow s$$
$$s \rightarrow s + L_p$$

$$\Longrightarrow \begin{aligned} C(s + L_p|s) &= \cos\sigma_0 - w(s)w'(s)\sin\sigma_0 \\ S(s + L_p|s) &= w^2(s)\sin\sigma_0 \end{aligned}$$

$$\boxed{w^2(s) = \beta(s) = \frac{S(s + L_p|s)}{\sin\sigma_0} = \frac{\mathbf{M}_{12}(s + L_p|s)}{\sin\sigma_0}}$$

Formula requires calculation of $S(s + L_p|s)$ at every value of *s* within lattice period

Previous formula requires one calculation of $C(s|s_i), \ S(s|s_i)$ for $s_i \leq s \leq s_i + L_p$ and any value of s_i

C4

Matrix algebra can be applied to simplify this result:

$$\mathbf{M}(s+L_p|s) = \mathbf{M}(s+L_p|s_i+L_p)\cdot\mathbf{M}(s_i+L_p|s)$$
$$= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s)\cdot[\mathbf{M}(s|s_i)\cdot\mathbf{M}^{-1}(s|s_i)]$$
$$= \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i)$$

$$\boxed{\mathbf{M}(s+L_p|s) = \mathbf{M}(s|s_i)\cdot\mathbf{M}(s_i+L_p|s_i)\cdot\mathbf{M}^{-1}(s|s_i)}$$

Using this result with the previous formula allows the transfer matrix to be calculated only once per period from any initial condition

Using:

$$\mathbf{M} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \qquad \mathbf{M}^{-1} = \begin{pmatrix} S' & -S \\ -C' & C \end{pmatrix}$$

Apply Wronskian condition:

$$\det \mathbf{M} = 1$$

The matrix formula can be shown to the equivalent to the previous one

Methodology applied in: Lund, Chilton, and Lee, PRSTAB **9** 064201 (2006) to construct a fail-safe iterative matched envelope including space-charge C5

S7: Hill's Equation: The Courant-Snyder Invariant and Single Particle Emittance

S7A: Introduction

Constants of the motion can simplify the interpretation of dynamics in physics

Desirable to identify constants of motion for Hill's equation for improved understanding of focusing in accelerators

Constants of the motion are not immediately obvious for Hill's Equation due to s-varying focusing forces related to $\kappa(s)$ can add and remove energy from the particle

- Wronskian symmetry is one useful symmetry
- Are there other symmetries?

/// Illustrative Example: Continuous Focusing/Simple Harmonic Oscillator

Equation of motion:

$$x'' + k_{\beta 0}^2 x = 0 \qquad k_{\beta 0}^2 = \text{const} > 0$$

Constant of motion is the well-know Hamiltonian/Energy:

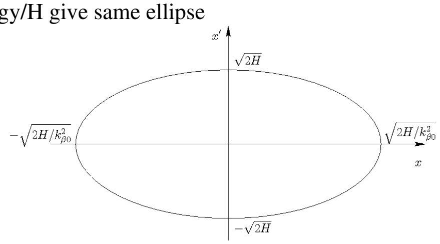
$$H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 \;\; = \text{const}$$

which shows that the particle moves on an ellipse in *x*-*x*' phase-space with:

Location of particle on ellipse set by initial conditions

All initial conditions with same energy/H give same ellipse

$$\text{Max/Min}[x] \Leftrightarrow x' = 0$$
$$\text{Max/Min}[x] = \pm\sqrt{2H/k_{\beta 0}^2}$$
$$\text{Max/Min}[x'] \Leftrightarrow x = 0$$
$$\text{Max/Min}[x'] = \pm\sqrt{2H}$$



///

Question:

For Hill's equation:

$$x'' + \kappa(s)x = 0$$

does a quadratic invariant exist that can aid interpretation of the dynamics?

Answer we will find:

Yes, the Courant-Snyder invariant

Comments:

Very important in accelerator physics

- Helps interpretation of linear dynamics

Named in honor of Courant and Synder who popularized it's use in Accelerator physics while co-discovering alternating gradient (AG) focusing in a single seminal (and very elegant) paper:

Courant and Snyder, *Theory of the Alternating Gradient Synchrotron*, Annals of Physics **3**, 1 (1958).

- Christofolos also understood AG focusing in the same period using a more heuristic analysis

Easily derived using phase-amplitude form of orbit solution

- Can be much harder using other methods

S7B: Derivation of Courant-Snyder Invariant

The phase amplitude method described in S6 makes identification of the invariant elementary. Use the phase amplitude form of the orbit:

$$\boxed{\begin{aligned} x(s) &= A_i w(s)\cos\psi(s) \\ x'(s) &= A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s) \end{aligned}}$$

$A_i,\ \psi_i = \psi(s_i)$ set by initial at $s = s_i$

where

$$w'' + \kappa(s) w - \frac{1}{w^3} = 0$$

Re-arrange the phase-amplitude trajectory equations:

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add the equations to obtain the Courant-Snyder invariant:

$$\boxed{\begin{aligned} \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= A_i^2(\cos^2\psi + \sin^2\psi) \\ &= A_i^2 = \text{const} \end{aligned}}$$

Comments on the Courant-Snyder Invariant:

- Simplifies interpretation of dynamics (will show how shortly)
- Extensively used in accelerator physics
- Quadratic structure in *x-x'* defines a rotated ellipse in *x-x'* phase space.
- Because $w^2\left(\frac{x}{w}\right)' = wx' - w'x$ the Courant-Snyder invariant can be alternatively expressed as:

$$\left(\frac{x}{w}\right)^2 + \left[w^2\left(\frac{x}{w}\right)'\right]^2 = \text{const}$$

Cannot be interpreted as a conserved energy!

The point that the Courant-Snyder invariant is *not* a conserved energy should be elaborated on. The equation of motion:

$$x'' + \kappa(s)x = 0$$

Is derivable from the Hamiltonian

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \implies \begin{aligned} \frac{d}{ds}x &= \frac{\partial H}{\partial x'} = x' \\ \frac{d}{ds}x' &= -\frac{\partial H}{\partial x} = -\kappa x \end{aligned} \implies x'' + \kappa x = 0$$

H is the energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 = T + V \qquad \begin{aligned} T &= \frac{1}{2}x'^2 &&= \text{Kinetic "Energy"} \\ V &= \frac{1}{2}\kappa x^2 &&= \text{Potential "Energy"} \end{aligned}$$

Apply the chain-Rule with $H = H(x,x';s)$:

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial x'}\frac{dx'}{ds}$$

Apply the equation of motion in Hamiltonian form:

$$\frac{d}{ds}x = \frac{\partial H}{\partial x'} \qquad \frac{d}{ds}x' = -\frac{\partial H}{\partial x}$$

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} \overbrace{- \frac{dx'}{ds}\frac{dx}{ds} + \frac{dx}{ds}\frac{dx'}{ds}}^{0} = \frac{\partial H}{\partial s} = \frac{1}{2}\kappa' x^2 \neq 0$$

$$\implies \boxed{H \neq \text{const}}$$

Energy of a "kicked" oscillator with $\kappa(s) \neq \text{const}$ is not conserved
Energy should not be confused with the Courant-Snyder invariant

/// Aside: Only for the special case of continuous focusing (i.e., a simple Harmonic oscillator) are the Courant-Snyder invariant and energy simply related:

Continuous Focusing: $\kappa(s) = k_{\beta 0}^2 = \text{const}$

$$\implies H = \frac{1}{2}x'^2 + \frac{1}{2}k_{\beta 0}^2 x^2 = \text{const}$$

w equation: $w'' + k_{\beta 0}^2 w - \frac{1}{w^3} = 0$

$$\implies w = \sqrt{\frac{1}{k_{\beta 0}}} = \text{const}$$

Courant-Snyder Invariant: $\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = \text{const}$

$$\begin{aligned} \implies \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= k_{\beta 0}x^2 + \frac{x'^2}{k_{\beta 0}} \\ &= \frac{2}{k_{\beta 0}}\left(\frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2\right) \\ &= \frac{2H}{k_{\beta 0}} = \text{const} \end{aligned}$$

///

Interpret the Courant-Snyder invariant:

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

by expanding and isolating terms quadratic terms in *x*-*x*' phase-space variables:

$$\left[\frac{1}{w^2} + w'^2\right] x^2 + 2[-ww']xx' + [w^2]x'^2 = A_i^2 = \text{const}$$

The three coefficients in [...] are functions of *w* and *w*' only and therefore are *functions of the lattice only* (not particle initial conditions). They are commonly called "Twiss Parameters" and are expressed denoted as:

$$\boxed{\begin{aligned} &\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 = \text{const} \\ &\gamma(s) \equiv \frac{1}{w^2(s)} + [w'(s)]^2 = \frac{1+\alpha^2(s)}{\beta(s)} \\ &\beta(s) \equiv w^2(s) \\ &\alpha(s) \equiv -w(s)w'(s) \end{aligned}}$$

$$\gamma\beta = 1 + \alpha^2$$

All Twiss "parameters" are specified by *w*(*s*)

Given *w* and *w*' at a point (*s*) any 2 Twiss parameters give the 3rd

The area of the invariant ellipse is:

Apply standard formulas from Analytic Geometry or calculate

$$\boxed{\text{Phase-Space Area} = \int_{\text{ellipse}} dx dx' = \frac{\pi A_i^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_i^2 \equiv \pi\epsilon}$$

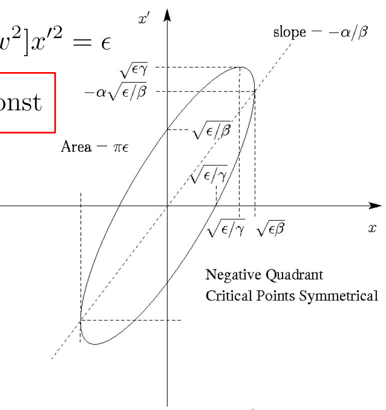
Where ϵ is the single-particle emittance:

Emittance is the area of the orbit in *x*-*x*' phase-space divided by π

$$[1/w^2 + w'^2]x^2 + 2[-ww']xx' + [w^2]x'^2 = \epsilon$$

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}}$$

See problem sets for critical point calculation



/// Aside on Notation: Twiss Parameters and Emittance Units:

Twiss Parameters:

Use of $\alpha,\ \beta,\ \gamma$ should not create confusion with kinematic relativistic factors

$\beta_b,\ \gamma_b$ are absorbed in the focusing function

Contextual use of notation unfortunate reality .... not enough symbols!

Notation originally due to Courant and Snyder, not Twiss, and might be more appropriately called "Courant-Snyder functions" or "lattice functions."

Emittance Units:

x has dimensions of length and *x*' is a dimensionless angle. So *x*-*x*' phase-space area has dimensions [[ϵ]] = length. A common choice of units is millimeters (mm) and milliradians (mrad), e.g.,

$$\epsilon = 10 \text{ mm-mrad}$$

The definition of the emittance employed is not unique and different workers use a wide variety of symbols. Some common notational choices:

$\pi\epsilon \to \epsilon \qquad \epsilon \to \varepsilon \qquad \epsilon \to E$

Write the emittance values in units with a π, e.g.,

$$\epsilon = 10.5\ \pi - \text{mm-mrad}$$

Use caution! Understand conventions being used before applying results! ///

Properties of Courant-Snyder Invariant:

The ellipse will rotate and change shape as the particle advances through the focusing lattice, but the instantaneous area of the ellipse ($\pi\epsilon = \text{const}$) remains constant.

The location of the particle on the ellipse and the size (area) of the ellipse depends on the initial conditions of the particle.

The orientation of the ellipse is independent of the particle initial conditions.

All particles move on nested ellipses.

Quadratic in the *x*-*x*' phase-space coordinates, but is *not* the transverse particle energy (which is not conserved).

S7C: Lattice Maps

The Courant-Snyder invariant helps us understand the phase-space evolution of the particles. Knowing how the ellipse transforms (twists and rotates without changing area) is equivalent to knowing the dynamics of a *bundle* of particles. To see this:

General s:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

Initial $s = s_i$

$$\gamma_i x_i^2 + 2\alpha_i x_i x_i' + \beta_i x_i'^2 = \epsilon$$

$$\beta_i \equiv \beta(s = s_i) \qquad x_i \equiv x(s = s_i)$$
$$\alpha_i \equiv \alpha(s = s_i) \qquad x_i' \equiv x'(s = s_i)$$
$$\gamma_i \equiv \gamma(s = s_i)$$

Apply the components of the transport matrix:

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \mathbf{M}(s|s_i) \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} C(s|s_i) & S(s|s_i) \\ C'(s|s_i) & S'(s|s_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

Invert 2x2 matrix and apply det **M** = 1 (Wronskian):

$$\Longrightarrow \quad \begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} S' & -S \\ -C' & C \end{bmatrix} \cdot \begin{bmatrix} x \\ x' \end{bmatrix} \qquad C \equiv C(s|s_i), \text{ etc.}$$

Insert expansion for $x_i, \ x_i'$ in the initial ellipse expression, collect factors of x^2, xx', and x'^2, and equate to general s ellipse expression:

$$\begin{aligned} &[\gamma_i S'^2 - 2\alpha_i S'C' + \beta_i C'^2]x^2 \\ +2&[-\gamma_i SS' + \alpha_i(CS' + SC') - \beta_i CC']xx' \\ +&[\gamma_i S^2 - 2\alpha_i SC + \beta_i C^2]x'^2 \\ &= \gamma x^2 + 2\alpha x x' + \beta x'^2 \end{aligned}$$

Collect coefficients of x^2, xx', and x'^2 and summarize in matrix form:

$$\boxed{\begin{bmatrix} \gamma \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} S'^2 & -2C'S' & C'^2 \\ -SS' & CS' + SC' & -CC' \\ S^2 & -2CS & C^2 \end{bmatrix} \cdot \begin{bmatrix} \gamma_i \\ \beta_i \\ \alpha_i \end{bmatrix}}$$

This result can be applied to illustrate how a bundle of particles will evolve from an initial location in the lattice subject to the linear focusing optics in the machine using only principal orbits C, S, C', and S'

Principal orbits will generally need to be calculated numerically

- Intuition can be built up using simple analytical results (hard edge etc)

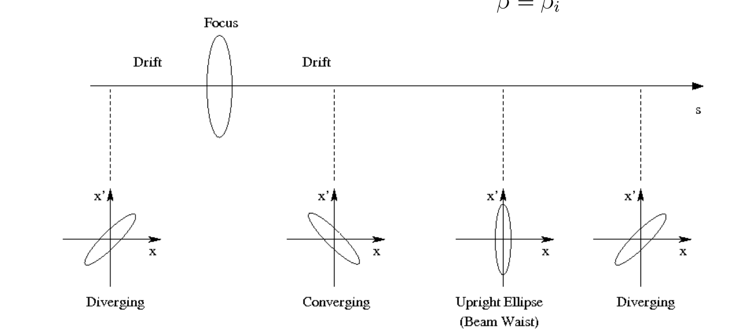
/// Example: Ellipse Evolution in a simple kicked focusing lattice

Drift:

$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix} \qquad \begin{aligned} \gamma &= \gamma_i \\ \alpha &= -\gamma_i(s - s_i) + \alpha_i \\ \beta &= \gamma_i(s - s_i)^2 - 2\alpha_i(s - s_i) + \beta_i \end{aligned}$$

Thin Lens:
focal length f

$$\begin{bmatrix} C & S \\ C' & S' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \qquad \begin{aligned} \gamma &= \gamma_i + 2\alpha_i/f + \beta_i/f^2 \\ \alpha &= -\beta_i/f + \alpha_i \\ \beta &= \beta_i \end{aligned}$$



For further examples of phase-space ellipse evolutions in standard lattices, see: S6G ///

S8: Hill's Equation: The Betatron Formulation of the Particle Orbit and Maximum Orbit Excursions S8A: Formulation

The phase-amplitude form of the particle orbit analyzed in S6 of

$$\boxed{x(s) = A_i w(s) \cos\psi(s) = \sqrt{\epsilon}\, w(s) \cos\psi(s)} \qquad [[w]] = (\text{meters})^{1/2}$$

is not a unique choice. Here, w has dimensions sqrt(meters), which can render it inconvenient in applications. Due to this and the utility of the Twiss parameters used in describing orientation of the phase-space ellipse associated with the Courant-Snyder invariant (see: S7) on which the particle moves, it is convenient to define an alternative, Betatron representation of the orbit with:

$$x(s) = \sqrt{\epsilon}\sqrt{\beta(s)} \cos\psi(s)$$

Betatron function: $\beta(s) \equiv w^2(s)$

Emittance: $\epsilon \equiv A_i^2 = \text{const}$

Phase: $\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})} = \psi_i + \Delta\psi(s)$

The betatron function has dimensions [[β]] = meters

Comments:

Use of the symbol β for the betatron function does not result in confusion with relativistic factors such as β_b since the context of use will make clear

- Relativistic factors often absorbed in lattice focusing function and do not directly appear in the dynamical descriptions

The change in phase $\Delta\psi$ is the same for both formulations:

$$\Delta\psi(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} = \int_{s_i}^{s} \frac{d\tilde{s}}{\beta(\tilde{s})}$$

From the equation for *w*:

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s+L_p) = w(s) \qquad w(s) > 0$$

the betatron function is described by:

$$\boxed{\begin{array}{l} \frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + \kappa(s)\beta^2(s) = 1 \\ \beta(s+L_p) = \beta(s) \qquad \beta(s) > 0 \end{array}}$$

The betatron function represents, analogously to the *w*-function, a special function defined by the periodic lattice

The equation is still nonlinear and must generally be solved numerically

S8B: Maximum Orbit Excursions

From the orbit equation

$$x = \sqrt{\epsilon\beta}\cos\psi$$

the maximum and minimum possible particle excursions occur where:

$$\cos\psi = +1 \quad \longrightarrow \quad \text{Max}[x] = \sqrt{\epsilon\beta(s)} = \sqrt{\epsilon}w(s)$$

$$\cos\psi = -1 \quad \longrightarrow \quad \text{Min}[x] = -\sqrt{\epsilon\beta(s)} = -\sqrt{\epsilon}w(s)$$

Thus, the max radial extent of *all* particle oscillations $\text{Max}[x] \equiv x_m$ in the beam distribution occurs for the particle with the max single particle emittance since the particles move on nested ellipses:

$$\boxed{\begin{array}{l} \text{Max}[\epsilon] \equiv \epsilon_m \\ x_m(s) = \sqrt{\epsilon_m\beta(s)} = \sqrt{\epsilon_m}w(s) \end{array}}$$

In terms of Twiss parameters:

$$\boxed{\begin{array}{l} x_m = \sqrt{\epsilon_m}w = \sqrt{\epsilon_m\beta} \\ x'_m = \sqrt{\epsilon_m}w' = -\sqrt{\frac{\epsilon_m}{\beta}}\alpha \end{array}}$$

Assumes sufficient numbers of particles to populate all possible phases

x_m corresponds to the min possible machine aperture to prevent particle losses

- Practical aperture choice influenced by: resonance effects due to nonlinear applied fields, space-charge, scattering, finite particle lifetime, ...

From:

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s+L_p) = w(s) \qquad w(s) > 0$$

We immediately obtain an equation for the maximum locus (envelope) of radial particle excursions $x_m = \sqrt{\epsilon_m}w$ as:

$$\boxed{\begin{array}{l} x''_m(s) + \kappa(s)x_m(s) - \frac{\epsilon_m^2}{x_m^3(s)} = 0 \\ x_m(s+L_p) = x_m(s) \qquad x_m(s) > 0 \end{array}}$$

Comments:

Equation is analogous to the statistical envelope equation derived by J.J. Barnard in the Intro Lectures when a space-charge term is added and the max single particle emittance is interpreted as a statistical emittance

- correspondence will become more concrete in later lectures

This correspondence will be developed more extensively in later lectures on Transverse Centroid and Envelope Descriptions of Beam Evolution and Transverse Equilibrium Distributions

S9: Momentum Spread Effects and Bending
S9A: Formulation

Except for brief digressions in S1 and S4, we have concentrated on particle dynamics where all particles have the design longitudinal momentum:

$$\boxed{p_s = m\gamma_b\beta_b c = \text{const}}$$

Realistically, there will always be a finite spread of particle momentum within a beam slice, so we take:

$$\boxed{\begin{aligned} &p_s = p_0 + \delta p \\ &p_0 \equiv m\gamma_b\beta_b c = \text{Design Momentum} \\ &\delta p \equiv \text{Off Momentum} \end{aligned}}$$

Typical values of momentum spread in a beam with a single species of particles with conventional sources and accelerating structures:

$$\frac{|\delta p|}{p_0} \sim 10^{-2} \rightarrow 10^{-6}$$

The spread of particle momentum can modify particle orbits, particularly when dipole bends are present since the bend radius depends strongly on the particle momentum

To better understand this effect, we analyze the particle equations of motion with leading-order momentum spread (see: S1H) effects retained:

$$x''(s) + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa_x(s)}{(1+\delta)^n}\right] x(s) = \frac{\delta}{1+\delta}\frac{1}{R(s)}$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

$R(s) =$ Local Bend Radius for design momentum p_0 ($R \rightarrow \infty$ in straight sections)

Magnetic Dipole Bend

$$\frac{1}{R(s)} = \frac{B_y^a|_{\text{dipole}}}{[B\rho]} \qquad [B\rho] = \frac{p_0}{q}$$

$$\delta \equiv \frac{\delta p}{p_0}$$

$\kappa_{x,y} =$ Focusing Functions (using design momentum)

$$n = \begin{cases} 1, & \text{Magnetic Quadrupoles} \\ 2, & \text{Solenoids, Electric Quadrupoles} \end{cases}$$

Neglects:

- Space-charge: $\phi \rightarrow 0$
- Nonlinear applied focusing: $\mathbf{E}^a$, $\mathbf{B}^a$ contain only linear focus terms
- Acceleration: $p_0 = m\gamma_b\beta_b = \text{const}$

In the equations of motion, it is important to understand that B_y^a of the magnetic bends are set from the radius *R* required by the design particle orbit (see: S1 for details)

- Equations must be modified slightly for electric bends (see S1)
- *y*-plane bends also require modification

The focusing strengths are defined with respect to the design momentum:

$$\kappa_x = \begin{cases} \frac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \frac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

γ_b, β_b calculated from p_0

Terms in the equations of motion associated with momentum spread (δ) can be lumped into two classes:

1) Chromatic -- Associated with Focusing
2) Dispersive -- Associated with Dipole Bends

S9B: Chromatic Effects

Present in both *x*- and *y*-equations of motion and result from applied focusing strength changing with deviations in momentum:

$$x''(s) + \frac{\kappa_x(s)}{(1+\delta)^n} x(s) = 0$$

$$y''(s) + \frac{\kappa_y(s)}{(1+\delta)^n} y(s) = 0$$

$R \rightarrow \infty$ to neglect bending terms

$\kappa_{x,y} =$ Focusing Functions with γ_b, β_b calculated from p_0

Generally of lesser importance (smaller corrections) relative to dispersive terms (S9C) *except* where the beam is focused onto a target (small spot) or when momentum spreads are large

Lectures by J.J. Barnard on Heavy Ion Fusion and Final Focusing will overview consequences of chromatic effects in final focus optics

S9C: Dispersive Effects

Present in only the x-equation of motion and result from bending. Neglecting chromatic terms:

$$x''(s) + \underbrace{\left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta} + \kappa_x(s)\right] x(s)}_{\text{Term 1}} = \underbrace{\frac{\delta}{1+\delta}\frac{1}{R(s)}}_{\text{Term 2}}$$

Particles are bent at different radii when the momentum deviates from the design value ($\delta \neq 0$) leading to changes in the particle orbit

Dispersive terms contain the bend radius R

Generally, the bend radii R are large and δ is small, and we can take to leading order:

Term 1: $\left[\frac{1}{R^2}\frac{1-\delta}{1+\delta} + \kappa_x\right] x \simeq \kappa_x x$

Term 2: $\frac{\delta}{1+\delta}\frac{1}{R} \simeq \frac{\delta}{R}$

The equations of motion then become:

$$\boxed{\begin{aligned} x''(s) + \kappa_x(s)x(s) &= \frac{\delta}{R(s)} \\ y''(s) + \kappa_y(s)y(s) &= 0 \end{aligned}}$$

The y-equation is not changed from the usual Hill's Equation

Generally, the x-equation is solved for periodic lattices by exploiting the linear structure of the equation and linearly resolving:

$$\boxed{\begin{aligned} &x(s) = x_h(s) + x_p(s) \\ &x_h \equiv \text{Homogeneous Solution} \\ &x_p \equiv \text{Particular Solution} \end{aligned}}$$

where x_h is the *general* solution to the Hill's Equation:

$$\boxed{x_h''(s) + \kappa_x(s)x_h(s) = 0}$$

and x_p is the *periodic* solution to:

$$\boxed{\begin{aligned} &x_p = \delta \cdot D \qquad && D''(s) + \kappa_x(s)D(s) = \frac{1}{R(s)} \\ &D \equiv \text{Disperson Function} && D(s+L_p) = D(s) \end{aligned}}$$

This convenient resolution of the orbit $x(s)$ can *always* be made because the homogeneous solution will be adjusted to match any initial condition

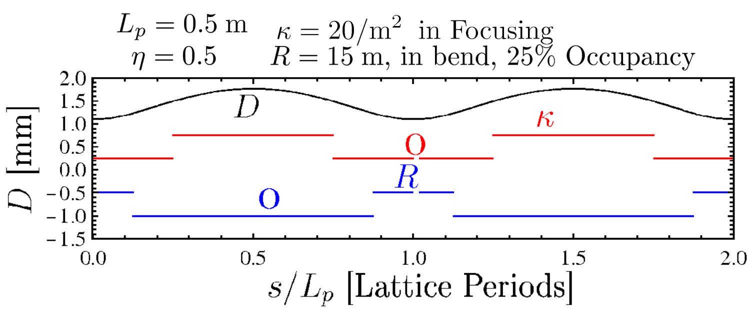
Note that x_p provides a measure of the offset of the particle orbit relative to the design orbit resulting from a small deviation of momentum (δ)

$x(s) = 0$ defines the design orbit

$[[D]] = \text{meters}$

$\delta \cdot D = \text{Orbit offset in meters}$

/// Example: Simple piecewise constant focusing and bending lattice



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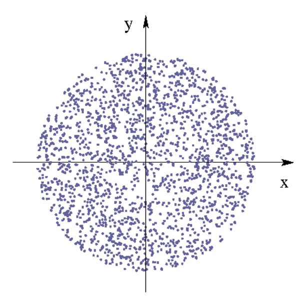
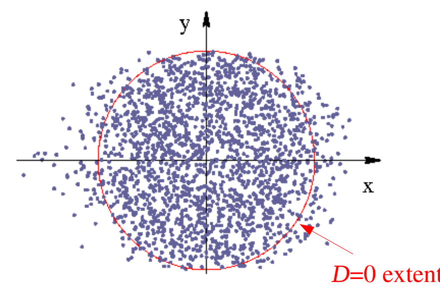
/// Example: Dispersion broadens the x-distribution

Uniform Bundle of particles D = 0



Same Bundle of particles D nonzero

Gaussian distribution of momentum spread distorts the x-y distribution extents in x but not in y



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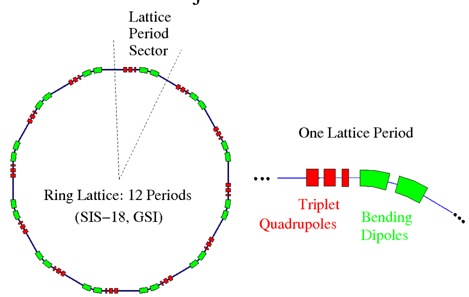
Many rings are designed to focus the dispersion function $D(s)$ to small values in straight sections even though the lattice has strong bends

- Desirable since it allows smaller beam sizes at locations near where $D = 0$ and these locations can be used to insert and extract (kick) the beam into and out of the ring with minimal losses
  - Since average value of D is dictated by ring size and focusing strength (see example next page) this variation in values can lead to D being larger in other parts of the ring
- Quadrupole triplet focusing lattices are often employed in rings since the optics allows sufficient flexibility to tune D while simultaneously allowing particle phase advances to also be adjusted



/// Example: Continuous Focusing in a Continuous Bend

$$\kappa_x(s) = k_{\beta 0}^2 = \text{const}$$
$$R(s) = R = \text{const}$$

Dispersion equation becomes:

$$D'' + k_{\beta 0}^2 D = \frac{1}{R}$$

With solution:

$$D = \frac{1}{k_{\beta 0}^2 R} = \text{const}$$

From this result we can crudely estimate the average value of the dispersion function in a ring with periodic focusing by taking:

$$R = \text{Avg Radius Ring}$$
$$L_p = \text{Lattice Period (Focusing)}$$
$$\sigma_{0x} = x\text{-Plane Phase Advance}$$

$$\Longrightarrow \quad k_{\beta 0} \sim \frac{\sigma_0}{L_p} \qquad \Longrightarrow \quad D \sim \frac{L_p^2}{\sigma_0^2 R}$$

///

S10: Acceleration and Normalized Emittance
S10A: Introduction

If the beam is accelerated longitudinally in a linear focusing channel, the x-particle equation of motion (see: S1 and S2) is:

$$\boxed{x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}}$$

Analogous equation holds in y

Neglects:

- Nonlinear applied focusing fields
- Momentum spread effects

Comments:

- $\gamma_b,\ \beta_b$ are regarded as prescribed functions of s set by the acceleration schedule of the machine
- Variations in $\gamma_b,\ \beta_b$ due to acceleration must be included in and/or compensated by adjusting the strength of the optics via $\kappa_x,\ \kappa_y$
  - Scaling different for electric and magnetic optics (see: S2)

Comments Continued:

- In typical accelerating systems, changes in $\gamma_b\beta_b$ are slow and the fractional changes in the orbit induced by acceleration are small
  - Exception near an injector since the beam is often not yet energetic
- The acceleration term:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} > 0$$

will act to damp particle oscillations (see following slides for motivation)

Even with acceleration, we will find that there is a Courant-Snyder invariant (normalized emittance) that is valid in an analogous context as in the case without acceleration provided phase-space coordinates are chosen to compensate for the damping of particle oscillations

Acceleration Factor: Characteristics of Relativistic Factor

$$\gamma_b\beta_b \simeq \begin{cases} \gamma_b, & \text{Relativistic Limit} \\ \beta_b, & \text{Nonrelativistic Limit} \end{cases} \qquad \gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}}$$

Beam/Particle Kinetic Energy:

$$\mathcal{E}_b(s) = (\gamma_b - 1)mc^2 = \text{Beam Kinetic Energy}$$

Function of s specified by Acceleration schedule for transverse dynamics
See S11 for calculation of $\mathcal{E}_b$ and $\gamma_b\beta_b$ from longitudinal dynamics
and J.J. Barnard lectures on Longitudinal Dynamics

Approximate energy gain from average gradient:

$$\mathcal{E}_b \simeq \mathcal{E}_i + G(s - s_i) \qquad \begin{aligned} \mathcal{E}_i &= \text{const} = \text{Initial Energy} \\ G &= \text{const} = \text{Average Gradient} \end{aligned}$$

Real energy gain will be rapid when going through discreet acceleration gaps

$$\mathcal{E}_b \simeq \begin{cases} \gamma_b mc^2, & \text{Relativistic Limit}, \gamma_b \gg 1 \\ \frac{1}{2}m\beta_b^2c^2, & \text{Nonrelativistic Limit}, |\beta_b| \ll 1 \end{cases}$$

Identify relativistic factor with average gradient energy gain:

Relativistic Limit: $\gamma_b \gg 1$

$$\gamma_b \simeq \frac{\mathcal{E}_b}{mc^2} = \frac{\mathcal{E}_i}{mc^2} + \frac{G}{mc^2}(s - s_i)$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} \simeq \frac{1}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s} \sim \frac{1}{s}$$

Nonrelativistic Limit: $|\beta_b| \ll 1$

$$\beta_b \simeq \sqrt{2\frac{\mathcal{E}_b}{mc^2}} = \sqrt{2\frac{\mathcal{E}_i}{mc^2} + 2\frac{G}{mc^2}(s - s_i)}$$

$$\Longrightarrow \quad \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{1/2}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s} \sim \frac{1}{2s}$$

Expect Relativistic and Nonrelativistic motion to have similar solutions
- Parameters for each case will often be quite different

/// Aside: Acceleration and Continuous Focusing Orbits with $\kappa_x = k_{\beta 0}^2 = \text{const}$

Assume relativistic motion and negligible space-charge:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\gamma_b'}{\gamma_b} = \frac{1}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s} \qquad \frac{\partial\phi}{\partial x} \simeq 0$$

Then the equation of motion reduces to:

$$x'' + \frac{1}{\left(\frac{\mathcal{E}_i}{G} - s_i\right) + s}x' + k_{\beta 0}^2 x = 0$$

This equation is the equation of a Bessel Function of order zero:

$$\frac{d^2x}{d\xi^2} + \frac{1}{\xi}\frac{dx}{d\xi} + x = 0 \qquad \xi = k_{\beta 0}s + k_{\beta 0}\left(\frac{\mathcal{E}_i}{G} - s_i\right)$$

$$\begin{aligned} x &= C_1 J_0(\xi) + C_2 Y_0(\xi) \\ x' &= -C_1 k_{\beta 0} J_1(\xi) - C_2 k_{\beta 0} Y_1(\xi) \end{aligned} \qquad \begin{aligned} &C_1 = \text{const} \quad C_2 = \text{const} \\ &J_n = \text{Order } n \text{ Bessel Func (1st kind)} \\ &Y_n = \text{Order } n \text{ Bessel Func (2nd kind)} \end{aligned}$$

Solving for the constants in terms of the particle initial conditions:

$$\begin{bmatrix} x_i \\ x_i' \end{bmatrix} = \begin{bmatrix} J_0(\xi_i) & Y_0(\xi_i) \\ -k_{\beta 0}J_1(\xi_i) & -k_{\beta 0}Y_1(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$\begin{aligned} x_i &\equiv x(s = s_i) \\ x_i' &\equiv x'(s = s_i) \end{aligned} \qquad \xi_i \equiv k_{\beta 0}\frac{\mathcal{E}_i}{G} = \xi(s = s_i)$$

Invert matrix to solve for constants in terms of initial conditions:

$$\Longrightarrow \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} -k_{\beta 0}Y_1(\xi_i) & -Y_0(\xi_i) \\ k_{\beta 0}J_1(\xi_i) & J_0(\xi_i) \end{bmatrix} \cdot \begin{bmatrix} x_i \\ x_i' \end{bmatrix}$$

$$\Delta \equiv k_{\beta 0}[Y_0(\xi_i)J_1(\xi_i) - J_0(\xi_i)Y_1(\xi_i)]$$

Comments:

Bessel functions behave like damped harmonic oscillators
- See any texts on Mathematical Physics or Applied Mathematics

Nonrelativistic limit solution is *not* described by a Bessel Function solution
- Properties of solution will be similar though (similar special function)
- The coefficient in the damping term $\propto x'$ has a factor of 2 difference, preventing exact Bessel function form

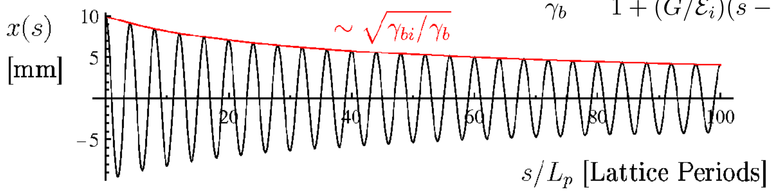
Using this solution, plot the orbit for (contrived parameters for illustration only):

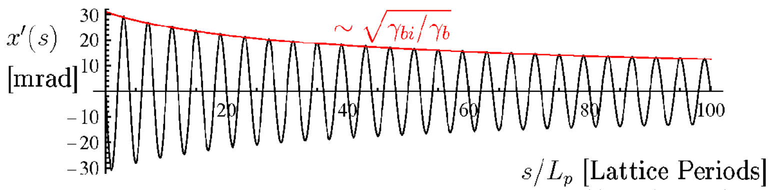
$$k_{\beta 0} = \frac{\sigma_0}{L_p} \qquad \sigma_0 = 90^\circ/\text{Period} \qquad \mathcal{E}_i = 1000 \text{ MeV}$$

$$L_p = 0.5 \text{ m} \qquad G = 100 \text{ MeV/m}$$

$$x(0) = 10 \text{ mm} \qquad s_i = 0$$

$$x'(0) = 0 \text{ mrad} \qquad \frac{\gamma_{bi}}{\gamma_b} = \frac{1}{1 + (G/\mathcal{E}_i)(s - s_i)}$$





Solution shows damping: phase volume scaling $\sim 1/(\gamma_b\beta_b) \simeq 1/\gamma_b$ ///

S10B: Transformation to Normal Form

"Guess" transformation to apply motivated by conjugate variable arguments (see: J.J. Barnard, Intro. Lectures)

$$\boxed{\tilde{x} \equiv \sqrt{\gamma_b\beta_b}\,x}$$

Then:

$$x = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}$$

$$x' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}' - \frac{1}{2}\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}$$

$$x'' = \frac{1}{\sqrt{\gamma_b\beta_b}}\tilde{x}'' - \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)^{3/2}}\tilde{x}' + \left[\frac{3}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^{5/2}} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)^{3/2}}\right]\tilde{x}$$

The inverse phase-space transforms will also be useful later:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}\,x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}\,x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b)}}x$$

Applying these results, the particle *x*- equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[\kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}\right]\tilde{x} = -\frac{q}{m\gamma_b^2\beta_b c^2}\frac{\partial\phi}{\partial\tilde{x}}$$

Note:

Factor of $\gamma_b\beta_b$ difference from untransformed expression in the space-charge coupling coefficient

It is instructive to also transform the Possion equation associated with the space-charge term:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

Transform:

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\right)\left(\frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}}\right) = \gamma_b\beta_b\frac{\partial^2}{\partial\tilde{x}^2}$$

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial\tilde{y}}{\partial y}\frac{\partial}{\partial\tilde{y}}\right)\left(\frac{\partial\tilde{y}}{\partial y}\frac{\partial}{\partial\tilde{y}}\right) = \gamma_b\beta_b\frac{\partial^2}{\partial\tilde{y}^2}$$

Using these results, Poisson's equation becomes:

$$\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\phi = -\frac{\rho}{\gamma_b\beta_b\epsilon_0}$$

Or defining a transformed potential $\tilde{\phi}$

$$\tilde{\phi} = \gamma_b\beta_b\phi$$

$$\left(\frac{\partial^2}{\partial\tilde{x}^2} + \frac{\partial^2}{\partial\tilde{y}^2}\right)\tilde{\phi} = -\frac{\rho}{\epsilon_0}$$

Applying these results, the *x*-equation of motion with acceleration becomes:

$$\tilde{x}'' + \left[\kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}\right]\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\tilde{\phi}}{\partial\tilde{x}}$$

Usual form of the space-charge coefficient with $\gamma_b^3\beta_b^2$ rather than $\gamma_b^2\beta_b$ is restored when expressed in terms of the transformed potential $\tilde{\phi}$

An additional step can be taken to further stress the correspondence between the transformed system with acceleration and the untransformed system in the absence of acceleration.
Denote an effective focusing strength:

$$\boxed{\tilde{\kappa}_x \equiv \kappa_x + \frac{1}{4}\frac{(\gamma_b\beta_b)'^2}{(\gamma_b\beta_b)^2} - \frac{1}{2}\frac{(\gamma_b\beta_b)''}{(\gamma_b\beta_b)}}$$

$\tilde{\kappa}_x$ incorporates acceleration terms beyond γ_b, β_b factors already included in the definition of κ_x (see: S2):

$$\kappa_x = \begin{cases} \frac{qG}{m\gamma_b\beta_b^2c^2}, & G = -\partial E_x^a/\partial x = \partial E_y^a/\partial y = \text{Electric Quad. Grad.} \\ \frac{qG}{m\gamma_b\beta_b c}, & G = \partial B_x^a/\partial y = \partial B_y^a/\partial x = \text{Magnetic Quad. Grad.} \\ \frac{qB_{z0}}{4m\gamma_b^2\beta_b^2c^2}, & B_{z0} = \text{Solenoidal Magnetic Field} \end{cases}$$

The transformed equation of motion with acceleration then becomes:

$$\boxed{\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\tilde{\phi}}{\partial\tilde{x}}}$$

The transformed equation with acceleration has the same form as the equation in the absence of acceleration. If space-charge is negligible ($\partial\phi/\partial\mathbf{x}_\perp \simeq 0$) we have:

Accelerating System | Non-Accelerating System

$$\tilde{x}'' + \tilde{\kappa}_x\tilde{x} = 0 \qquad \Longrightarrow \qquad x'' + \kappa_x x = 0$$

Therefore, *all previous analysis* on phase-amplitude methods and Courant-Snyder invariants associated with Hill's equation in *x*-*x*' phase-space can be immediately applied to $\tilde{x} - \tilde{x}'$ phase-space for an accelerating beam

$$\left(\frac{\tilde{x}}{\tilde{w}_x}\right)^2 + (\tilde{w}_x\tilde{x}' - \tilde{w}_x'\tilde{x})^2 = \tilde{\epsilon} = \text{const}$$

$$\tilde{w}_x'' + \tilde{\kappa}_x\tilde{w}_x - \frac{1}{\tilde{w}_x^3} = 0$$

$$\tilde{w}_x(s + L_p) = \tilde{w}_x(s)$$

$$\pi\tilde{\epsilon} = \text{Area traced by orbit} = \text{const in } \tilde{x}\text{-}\tilde{x}' \text{ phase-space}$$

Focusing field strengths need to be adjusted to maintain periodicity of $\tilde{\kappa}_x$ in the presence of acceleration
- Not possible to do exactly, but can be approximate for weak acceleration

S10C: Phase Space Relation Between Transformed and UnTransformed Systems

It is instructive to relate the transformed phase-space area in tilde variables to the usual *x*-*x*' phase area:

$$d\tilde{x} \otimes d\tilde{x}' = |J| dx \otimes dx'$$

where *J* is the Jacobian:

$$J \equiv \det\begin{bmatrix} \frac{\partial\tilde{x}}{\partial x} & \frac{\partial\tilde{x}}{\partial x'} \\ \frac{\partial\tilde{x}'}{\partial x} & \frac{\partial\tilde{x}'}{\partial x'} \end{bmatrix}$$

$$= \det\begin{bmatrix} \sqrt{\gamma_b\beta_b} & 0 \\ \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{\gamma_b\beta_b}} & \sqrt{\gamma_b\beta_b} \end{bmatrix} = \gamma_b\beta_b$$

Inverse transforms derived previously:

$$\tilde{x} = \sqrt{\gamma_b\beta_b}\,x$$

$$\tilde{x}' = \sqrt{\gamma_b\beta_b}\,x' + \frac{1}{2}\frac{(\gamma_b\beta_b)'}{\sqrt{(\gamma_b\beta_b)}}x$$

Thus:

$$\boxed{d\tilde{x} \otimes d\tilde{x}' = \gamma_b\beta_b\; dx \otimes dx'}$$

Based on this area transform, if we define the (instantaneous) phase space area of the orbit trance in *x*-*x*' to be $\pi\epsilon_x$ "regular emittance", then this emittance is related to the "normalized emittance" $\tilde{\epsilon}_x$ in $\tilde{x} - \tilde{x}'$ phase-space by:

$$\boxed{\begin{aligned} \tilde{\epsilon}_x &= \gamma_b\beta_b\epsilon_x \\ &\equiv \text{Normalized Emittance} \equiv \epsilon_{nx} \end{aligned}}$$

Factor $\gamma_b\beta_b$ compensates for acceleration induced damping in particle orbits
Normalized emittance is very important in design of lattices to transport accelerating beams
- Designs usually made assuming conservation of normalized emittance

Same result that J.J. Barnard motivated in the Intro. Lectures using alternative methods

S11: Accelerating Fields and Calculation of Changes in gamma*beta
S11A: Introduction

The transverse particle equation of motion with acceleration was derived in a Cartesian system by approximating (see: S1):

$$\frac{d}{dt}\left(m\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq \quad q\mathbf{E}^a_\perp + q\beta_b c\hat{\mathbf{z}}\times\mathbf{B}^a_\perp + qB^a_z\mathbf{v}_\perp\times\hat{\mathbf{z}} \quad - q\frac{1}{\gamma_b^2}\frac{\partial\phi}{\mathbf{x}_\perp}$$

using

$$m\frac{d}{dt}\left(\gamma\frac{d\mathbf{x}_\perp}{dt}\right) \simeq m\gamma_b\beta_b^2c^2\left[\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp\right]$$

to obtain:

$$\boxed{\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp\times\hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi}$$

To integrate this equation, we need the variation of β_b and $\gamma_b = 1/\sqrt{1-\beta_b^2}$ as a function of s. For completeness here, we briefly outline how this can be done by analyzing longitudinal equations of motion. More details can be found in JJ Barnard lectures on longitudinal dynamics.

S11B: Solution of Longitudinal Equation of Motion

Changes in $\gamma_b\beta_b$ are calculated from the longitudinal particle equation of motion:

$$\frac{d}{dt}\left(m\gamma\frac{dz}{dt}\right) \simeq \quad qE^a_z \quad - q(v_xB^a_y - v_yB^a_x) \quad - q\frac{\partial\phi}{\partial z}$$

Term 1   Term 2   Term 3   Neglect Rel to Term 2

Using steps similar to those in S1, we approximate terms:

$$\frac{dz}{dt} = v_z \simeq \beta_b c \qquad \gamma \simeq \gamma_b$$

$$\frac{d}{dt} \simeq \beta_b c\frac{d}{ds}$$

Term 1: $\frac{d}{dt}\left(\gamma\frac{dz}{dt}\right) \simeq c^2\beta_b(\gamma_b\beta_b)'$

Term 2: $\frac{q}{m}E^a_z \simeq -\frac{q}{m}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}$

ϕ^a is a quasi-static approximation accelerating potential (see next pages)

Term 3: $-q(v_xB^a_y - v_yB^a_x) = -q\left(\frac{dx}{dt}B^a_y - \frac{dy}{dt}B^a_x\right) \simeq 0$

Transverse magnetic fields typically only weakly change particle energy and terms can be neglected relative to others

The longitudinal particle equation of motion for γ_b, β_b then reduces to:

$$\boxed{\beta_b(\gamma_b\beta_b)' \simeq -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}}$$

Some algebra then shows that:

$$\gamma'_b = \left(\frac{1}{\sqrt{1-\beta_b^2}}\right)' = \gamma_b^3\beta_b\beta'_b$$

$$\Longrightarrow \quad \beta_b(\gamma_b\beta_b)' = \beta_b^2\gamma'_b + \gamma_b\beta_b\beta'_b$$
$$= (1+\gamma_b^2\beta_b^2)\gamma_b\beta_b\beta'_b = \gamma_b^3\beta_b\beta'_b$$
$$= \gamma'_b$$

Giving:

$$\boxed{\gamma'_b = -\frac{q}{mc^2}\left.\frac{\partial\phi^a}{\partial s}\right|_{x=y=0}}$$

Which can then be integrated to obtain:

$$\boxed{\gamma_b = -\frac{q}{mc^2}\phi^a(r=0, z=s) + \text{const}}$$

We denote the on-axis accelerating potential as:

$$\boxed{V(s) \equiv \phi^a(x = y = 0, z = s)}$$

Can represent RF or induction accelerating gap fields
See: J.J. Barnard lectures for more details

Using this and setting $\gamma_b(s = s_i) = \gamma_{bi}$ gives for the gain in axial kinetic energy $\mathcal{E}_b$ and corresponding changes in $\gamma_b,\ \beta_b$ factors:

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$

$$\gamma_b = 1 + \mathcal{E}_{bi}/(mc^2) \qquad \mathcal{E}_{bi} = (\gamma_{bi} - 1)mc^2$$

$$\beta_b = \sqrt{1 - 1/\gamma_b^2}$$

These equations can be solved for the consistent variation of $\gamma_b(s),\ \beta_b(s)$ to integrate the transverse equations of motion:

$$\mathbf{x}''_\perp + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}'_\perp = \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}^a_\perp + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}} \times \mathbf{B}^a_\perp + \frac{qB^a_z}{m\gamma_b\beta_b c}\mathbf{x}'_\perp \times \hat{\mathbf{z}} - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial \mathbf{x}_\perp}\phi$$

Nonrelativistic limit results

In the nonrelativistic limit:

$$\gamma_b \simeq 1 + \frac{1}{2}\beta_b^2 \qquad \beta_b^2 \ll 1 \qquad \mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2c^2$$

and the previous relativistic energy gain formulas reduce to:

$$\mathcal{E}_b \simeq \frac{1}{2}m\beta_b^2c^2 = q[V(s_i) - V(s)] + \mathcal{E}_{bi}$$

$$\gamma_b \simeq 1 \qquad \mathcal{E}_{bi} = \frac{1}{2}m\beta_b^2c^2$$

$$\beta_b = \sqrt{\frac{2\mathcal{E}_b}{mc^2}}$$

Using this result, in the nonrelativistic limit we can take in the transverse particle equation of motion:

$$\boxed{\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} \simeq \frac{\beta_b'}{\beta_b} = \frac{\mathcal{E}_b'}{\mathcal{E}_b} = -\frac{1}{2}\frac{qV'(s)}{q[V(s_i) - V(s)] + \mathcal{E}_{bi}}}$$

S11C: Longitudinal Solution via Energy Gain

An alternative analysis of the particle energy gain carried out in S11B can be illuminating. In this case we start from the exact Lorentz force equation with time as the independent variable for a particle moving in the full electromagnetic field:

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E} + q\vec{\beta}c \times \mathbf{B}$$

$$\mathbf{p} \equiv \gamma m\vec{\beta}c \qquad \gamma \equiv 1/\sqrt{1 - \vec{\beta}\cdot\vec{\beta}}$$

Dotting $mc\vec{\beta}$ into this equation:

$$mc\vec{\beta}\cdot\frac{d}{dt}(c\gamma\vec{\beta}) = qc\vec{\beta}\cdot\mathbf{E} + \underbrace{qc\vec{\beta}\cdot[c\vec{\beta} \times \mathbf{B}]}_{0}$$

$$\vec{\beta}\cdot\vec{\beta}\dot{\gamma} + \gamma\vec{\beta}\cdot\dot{\vec{\beta}} == \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}$$

and

$$\gamma \equiv (1 - \vec{\beta}\cdot\vec{\beta})^{-1/2}$$

$$\vec{\beta}\cdot\vec{\beta} = 1 - 1/\gamma^2$$

$$\vec{\beta}\cdot\dot{\vec{\beta}} = \dot{\gamma}/\gamma^3$$

Inserting these factors:

$$(1 - 1/\gamma^2)\dot{\gamma} + \dot{\gamma}/\gamma = \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}$$

or:

$$\boxed{\dot{\gamma} = \frac{q}{mc^2}\vec{\beta}\cdot\mathbf{E}}$$

Equivalently:

$$\boxed{\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\left[(\gamma - 1)mc^2\right] = qc\vec{\beta}\cdot\mathbf{E}}$$

- Only the electric field changes the kinetic energy of a particle

Taking: $\frac{d}{dt} = c\beta_z\frac{d}{ds}$ $\qquad \beta_z \simeq \beta \simeq \beta_b$, $\quad \gamma \simeq \gamma_b$

and approximating the axial electric field by the applied component then obtains

$$\boxed{\frac{d}{ds}\mathcal{E}_b = \frac{d}{ds} = \frac{d}{dt}\left[(\gamma - 1)mc^2\right] \simeq qE^a_z}$$

which is the longitudinal equation of motion analyzed in S11B.

S11D: Quasistatic Potential Expansion

In the quasistatic approximation, the accelerating potential can be expanded in the axisymmetric limit as:

See: J.J. Barnard, Intro Lectures; and Reiser, *Theory and Design of Charged Particle Beams*, (1994, 2008) Sec. 3.3.

$$\boxed{\phi^a = V(z) - \frac{1}{4}\frac{\partial^2}{\partial z^2}V(z)(x^2+y^2) + \frac{1}{64}\frac{\partial^4}{\partial z^4}V(z)(x^2+y^2)^2 + \cdots}$$

The longitudinal acceleration also result in a transverse focusing field

$$\mathbf{E}^a_\perp = \mathbf{E}^a_\perp|_{\rm foc} - \frac{\partial \phi^a}{\partial \mathbf{x}_\perp}$$

$$\mathbf{E}^a_\perp|_{\rm foc} = \text{Fields from Any Applied Focusing Optics}$$

$$-\frac{\partial \phi^a}{\partial \mathbf{x}_\perp} \simeq \frac{1}{2}\frac{\partial^2}{\partial z^2}V(z)\mathbf{x}_\perp = \text{Focusing Field from Acceleration}$$

Results can be used to cast acceleration terms in more convenient forms. See J.J. Barnard, Intro. Lectures for more details.

Einzel lens focusing exploits accel/de-acell cycle to make AG focusing

These notes will be corrected and expanded for reference and future editions of US Particle Accelerator School and University of California at Berkeley courses:

"Beam Physics with Intense Space Charge"

"Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

by J.J. Barnard and S.M. Lund

Corrections and suggestions for improvements are welcome. Contact:

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